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Low-frequency sound radiation and generation due to the interaction of unsteady flow with a jet pipe

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In this paper we examine the low-frequency sound radiated when various types of unsteady flow interact with a jet pipe. In each case we solve the problem exactly by the Wiener–Hopf technique, producing results valid for arbitrary internal and external Mach numbers and temperatures, discuss the importance of a Kutta condition at the duct exit, and provide an interpretation, in elementary terms, of the radiated sound field using the Lighthill acoustic analogy. A central feature is that the solutions are always obtained subject to a causality requirement, regardless of whether or not a Kutta condition is imposed at the pipe lip.

When low-frequency sound propagates down the jet pipe, little of it reaches the far field, and the major disturbance outside the pipe is that associated with the jet instability waves. At subsonic jet speeds and low-enough Strouhal number these waves transport kinetic energy at a rate precisely balancing the loss of acoustic energy from the pipe, resulting in a net attenuation of the sound power. For supersonic jet conditions a further wave motion, the unsteady-flow counterpart of the steady wave structure of an imperfectly expanded jet, is present in addition to the instability wave. We use the Lighthill acoustic analogy to show that, for high-enough jet Mach number and temperature, the sound radiation is caused largely by quadrupole sources arising from the jet instability waves. An alternative interpretation uses the acoustic analogy incorporating a mean flow due to Dowling, Ffowcs Williams and Goldstein, and expresses the far-field sound as the sum of contributions from monopoles and dipoles distributed over the duct exit. The directivity and power of the calculated far-field sound are in good agreement with experiments.

We also calculate the sound scattered by the jet pipe when there is an incident external sound field, and show a previously published result to be in error. In general, the flow phenomena produced by internal and external incident sound fields are similar. Finally, we discuss the effects of nozzle contraction. We find that the radiated sound field is little changed in character, but that the reflection properties of the nozzle may be drastically altered.

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1. Introduction

In this paper we examine the interaction between a number of types of unsteady flow and a jet pipe. The motivation behind this study was the so-called 'excess-noise' problem on jet engines. It has been found that, when the noise of an engine is measured statically, it is somewhat greater than would have been predicted on the basis of tests on model jets. This discrepancy is even greater in flight and has been the subject of a great deal of research (Bryce 1979). In this paper we model some of the possible mechanisms of excess noise: the transmission of internally generated noise out of the jet pipe to the far field, the scattering of external sound fields by the jet pipe, and the convection of turbulence past the end of the jet pipe. We consider only the low-frequency limit, but unlike many other authors we allow the mean flows both outside and inside the pipe to have arbitrary Mach numbers and temperatures. This is important, since for the conditions of interest (typically jet Mach number 0·8, internal-to-external temperature ratio 2·5) the effects of flow may be considerable. For example, Goldstein (1975) shows that placing a low-frequency acoustic source inside a jet flow has a dramatic effect on the field shape of the radiated sound.

The problem we solve first is the propagation of acoustic waves out of the jet pipe to the far field; in this as in the other problems we idealize the propulsion nozzle as a semi-infinite rigid cylindrical pipe. The mean flow outside the pipe consists of a uniform semi-infinite jet bounded by a vortex sheet. We confine the discussion to low frequencies, where the incident sound field in the pipe is in the form of plane waves. This problem in the absence of a mean flow was first solved by Levine & Schwinger (1948) using the Wiener–Hopf method. Their solution was extended to include the *same* uniform mean flow both inside and outside the pipe by Carrier (1956). The first attempts to include the effects of different mean flows inside and outside the pipe were made by Mani (1973) and Savkar (1975), for plane and circular pipe geometry respectively, who used an approximate method. The exact solution to the circular-pipe problem was found by Munt (1977) who, again using the Wiener–Hopf technique, allowed for arbitrary internal and external Mach numbers and temperatures and obtained field shapes for the radiated sound in excellent agreement with experiments. Munt (1982*a*, *b*) later extended his work to calculate both the amplitude of the sound reflected back up the pipe (1982*a*) and to examine the variation in the total power radiated with jet conditions (1982*b*). The power radiation has been studied experimentally by Bechert, Michel & Pfizenmaier (1977), who observed that the power radiated to the far field could be substantially less than the net power flow along the pipe, so that there was a net loss of acoustic energy. Munt's (1982*b*) paper is consistent with these results, as is the work of Howe (1979), who has studied the sound-transmission problem in the low-Mach-number, low-frequency approximation. Bechert (1979) also explains this net power loss using a simple theory similar to that of the present work, only at very low Mach numbers.

In this paper, we use the low-frequency asymptote of Munt's theory to obtain simple expressions for the sound radiated to the far field, that reflected back up the jet pipe, and for the unsteady motion of the jet column. The latter consists mainly of a spatially growing instability wave. This is an important feature of all problems involving the interaction of unsteady flows and jet pipes. In the limit of vanishingly low frequency this instability wave grows only very slowly and is convected with the mean flow. In addition to the usual low-frequency limit we also discuss the case where the jet is very hot compared with its surroundings, so that, as it were, it is hotter than it is acoustically compact. Here there is a dramatic change in the nature of the radiated sound field, similar to that found by Dowling, Ffowcs Williams & Goldstein (1978) in their study of jet noise. In Munt's paper it is assumed that the sound radiated is causally related to the incident sound field and that a Kutta condition is obeyed at the duct exit. We discuss the effect of relaxing the Kutta condition while still insisting on causality, and establish that the jet instability wave can then be made to vanish. In that case, there is no loss of acoustic energy, and all the power in the incident wave is reflected back up the duct, apart from an $O(k^2a^2)$ fraction which is lost to the far field. We further use an idea of Howe (1979) to provide an alternative modelling of the instability waves.

Munt's solution only allows for subsonic jet speeds. We extend his theory to cover supersonic jet conditions, using concepts due to Morgan (1974), and show that an additional physical phenomenon is present at these speeds; the unsteady counterpart of the periodic steady wave structure of an imperfectly expanded supersonic jet.

Using methods similar to those of Munt, we determine the sound scattered when an external sound field is incident on the pipe. In the absence of a mean flow the solution is known (see e.g. Noble 1958) and may be deduced from that for incident internal sound by reciprocity arguments. There is no existing exact solution when a mean flow is present, the only published work being the approximate solution of Jacques (1975). We show that his solution is in error, although the scaling laws he deduces are substantially correct when the incident sound waves are due to some nearby aerodynamic disturbance.

We also discuss, in less detail, the generation of sound when turbulence is convected by the mean flow past the end of the jet pipe; a full description of the theory may be found in Cargill (1981). The published work on this low-frequency problem is limited to two cases. Leppington (1971) models the turbulence as non-convected quadrupoles, whose near field is scattered as sound by the end of the pipe, resulting in far-field sound levels which scale as the sixth power of the jet velocity. Crighton (1972) models the problem as the scattering of the energy of a jet instability wave by the pipe exit and finds the same overall scaling laws as Leppington. Related to these problems is work by Howe (1976) on the sound generated when vortices are convected past the trailing edge of a flat plate. He finds that the sound radiated depends critically on the imposition of a Kutta condition at the edge of the plate. When a Kutta condition is enforced and the vortices are convected with the mean flow, then no sound is radiated. In our study of the semi-infinite cylindrical pipe we find that a similar result holds.

A useful way of examining sound-radiation problems is by the use of acoustic analogies. These ascribe the sound radiation to monopole and dipole sources on bounding surfaces, and to quadrupole sources distributed throughout the flow field. We use two different acoustic analogies; that of Lighthill (1952) as reformulated by Ffowcs Williams & Hawkings (1969), which does not explicitly include the fluid-shielding effects of any mean flow, and that of Dowling *et al.* (1978), which does. In each case the source terms are determined using the lowest-order asymptotic-low frequency solutions for the flows in the jet and the pipe. We show that the sound fields determined in this way are precisely the same as those obtained exactly by the Wiener–Hopf method.

Thus far we have idealized the end of the jet pipe by a cylindrical pipe. On real engines the end of the jet pipe contracts to form a nozzle. We discuss the transmission of sound through such a nozzle, and the sound generated when entropy waves are convected through the contraction. We use the methods of Marble & Candel (1977) and Cumpsty & Marble (1977), who were concerned with variable-area ducts and the transmission of acoustic waves across turbines, respectively. Our results for the transmission problem are in good agreement with the recent experimental results of Bechert (1979) and our expressions for the sound generated by entropy waves are essentially the same as those obtained by Ffowcs Williams & Howe (1975) using another method.

Finally, we discuss the practical significance of our results and compare them with the limited experimental evidence.

2. Radiation of internal noise from a jet pipe with flow

In this section we consider the radiation of low-frequency internal noise from a cylindrical pipe with both internal and external flows. We first solve the problem for a subsonic jet in the low-frequency limit, subject to the condition that it satisfies a trailing-edge Kutta condition. Then, we discuss the implications of relaxing the Kutta condition, and finally modify the analysis to allow for supersonic jet conditions.

2.1. *Subsonic jet with Kutta condition*

The mathematics in this section largely follows the work of Munt (1977). For convenience, and to aid comparison with his papers, we use his notation. While this problem has been solved in some detail by Munt we repeat the steps in the mathematics since the analysis forms the basis for both the rest of this section and for § 3. The major difference between our analysis and Munt's is that we choose to work with pressure rather than velocity potential as the fundamental variable.

We consider a cylindrical semi-infinite rigid tube of radius a, from which issues a jet of density ρ_j, speed of sound c_j and velocity $U_j = Mc_j$, occupying the region $x > 0$, $r < a$. The jet and pipe are assumed to be immersed in an infinite region of velocity $U_0 = \alpha M c_j$, density $\rho_0 = \gamma\rho_j$ and speed of sound $c_0 = c_j/C$. We assume that α covers the range $0 \leqslant \alpha \leqslant 1$.

The non-dimensional quantities γ, C, α express the ratios of the mainstream-to-jet value of density, reciprocal of sound speed, and velocity. When the jet and mainstream are perfect gases with the same specific heats, we find that $\gamma = C^2$.

The waves in the pipe are assumed to have the time dependence $e^{i\omega t}$, and this factor is suppressed throughout the analysis. The equations satisfied by the pressure fluctuations in cylindrical co-ordinates are

$$\frac{\partial^2 p}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2}-\left(ik+M\frac{\partial}{\partial x}\right)^2 p = 0 \quad (r < a), \tag{2.1}$$

$$\frac{\partial^2 p}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p}{\partial r^2}-C^2\left(ik+M\alpha\frac{\partial}{\partial x}\right)^2 p = 0 \quad (r > a), \tag{2.2}$$

where $k = \omega/c_j$. From the assumption that the cylinder is rigid, one derives the boundary condition that the normal gradient of pressure vanishes on it:

$$\frac{\partial p}{\partial r}(a,\phi,x) = 0 \quad (x \leqslant 0). \tag{2.3}$$

The boundary conditions on the jet vortex layer are the continuity of pressure, so that

$$p(a^-,\phi,x) = p(a^+,\phi,x) \quad (x \geqslant 0), \tag{2.4}$$

and the kinematic condition of equal particle displacement on both sides of the vortex layer. Let $\eta(x,\phi)$ denote the displacement of the vortex layer from its mean position, $r = a$. Then this latter condition implies that η satisfies

$$\left.\begin{aligned} c_j^2\left(ik+M\frac{\partial}{\partial x}\right)^2\eta(x,\phi) &= -\frac{1}{\rho_j}\frac{\partial p}{\partial r}(a^-,\phi,x),\\ c_j^2\left(ik+\alpha M\frac{\partial}{\partial x}\right)^2\eta(x,\phi) &= -\frac{1}{\gamma\rho_j}\frac{\partial p}{\partial r}(a^+,\phi,x)\end{aligned}\right\} \quad (x > 0). \tag{2.5}$$

Two other conditions are important in determining the sound field: causality and the Kutta condition. Causality is defined to be the requirement that the sound field shall vanish for impulsive excitation before the source is switched on. As Jones & Morgan (1974) have shown, if a time-harmonic solution is used, this must then obey certain constraints on its behaviour in the lower half-plane for complex k. The Kutta condition concerns the requirement to be satisfied by the displacement of the vortex layer at the edge of the cylinder. The usual Kutta condition is that the layer should leave the end of the pipe with zero gradient. The solution found by Munt satisfies both causality and this Kutta condition. We shall later discuss solutions that are causal but do not satisfy a Kutta condition.

Accordingly we now require for our solution that

$$\frac{\partial \eta}{\partial x}(0^+, \phi) = 0. \tag{2.6}$$

We split the total field into two parts: an incident field which is assumed to be known and the additional term arising from its interaction with the pipe. We assume that the incident field has the form of an acoustic duct mode with

$$\left.\begin{aligned} p_i(r, \phi, x) &= \frac{J_m(j'_{mn} r/a)}{J_m(j'_{mn})} \exp\left[-i(\mu_{mn} x - m\phi)\right] \quad (r < a), \\ &= 0 \quad (r > a), \end{aligned}\right\} \tag{2.7}$$

which satisfies (2.1) and (2.3), and where

$$\mu_{mn} = \frac{[k^2 - (1 - M^2) j'^2_{mn}/a^2]^{\frac{1}{2}} - kM}{1 - M^2},$$

with $\mathscr{I}\mu < 0$. Here j'_{mn} is the nth zero of $dJ_m(y)/dy$, and $J_m(y)$ is the Bessel function of order m. Since the primary wave has the dependence $e^{im\phi}$, we further assume that the diffracted field has the same dependence.

To assist the analysis we assume that k has a negative imaginary part, so that any waves produced will decay as $x \to \pm\infty$. In particular we define $k = k_r + ik_i = |k|\, e^{-i\delta}$, where $0 < \delta < \pi$. At the end of the analysis we shall put $\delta = 0$ to obtain the solution for real ω.

We define the half-range Fourier transforms of any quantity ψ, say, by the formulae

$$\Psi^{\pm}(u) = \int_{-\infty}^{+\infty} \psi(x) \exp(+ikux)\, H(\pm x)\, dx, \tag{2.8}$$

where $H(x)$ is the unit step function with

$$H(x) = 1 \quad (x > 0),$$

$$H(x) = 0 \quad (x < 0).$$

The inverse of these transforms is given correspondingly by

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi(u)\, k \exp(-ikux)\, du, \tag{2.9}$$

where $\Psi(u) = \Psi^+(u) + \Psi^-(u)$.

After Fourier transformation the equations of motion (2.1) and (2.2) become

$$\left\{\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 v^2(u) - \frac{m^2}{r^2}\right\} P(k, r) = 0 \quad (r < a), \tag{2.10}$$

$$\left\{\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 w^2(u) - \frac{m^2}{r^2}\right\} P(k, r) = 0 \quad (r > a), \tag{2.11}$$

in which we have defined

$$v^2(u) = (1 - Mu)^2 - u^2, \tag{2.12a}$$

$$w^2(u) = (1 - M\alpha u)^2 - u^2. \tag{2.12b}$$

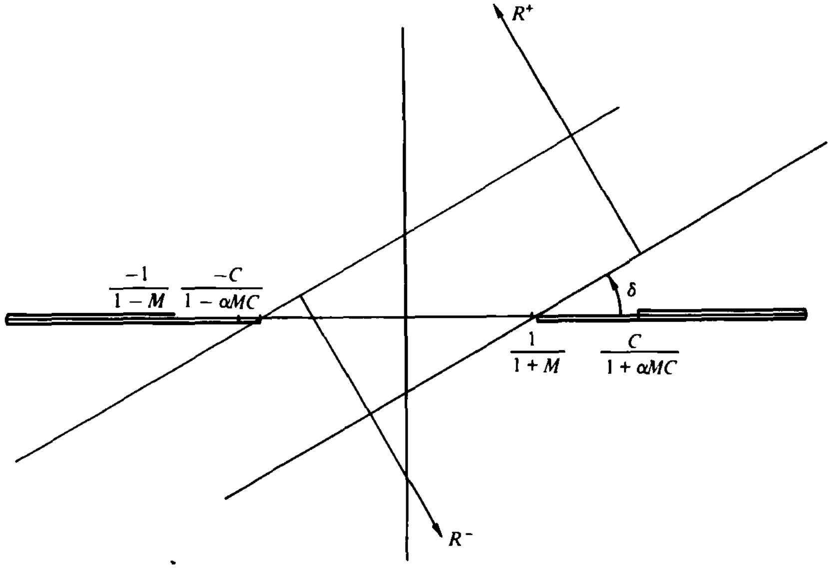

FIGURE 1. Positions of branch cuts and regions of regularity R^+, R^-, in the complex u-plane for subsonic flow.

The branches of w, v are taken to be those where $\mathscr{I}(v, w) < 0$ as $u \to +\infty$. The dependence on u of the transform $P(k, r, u)$ will sometimes be omitted, as in (2.10) and (2.11), while elsewhere it will be the dependence on (u, r) which is explicitly displayed. The branch cuts are taken to be from

$$u = \frac{1}{1+M} \text{ to } +\infty \quad \text{and} \quad \frac{-1}{1-M} \text{ to } -\infty \quad \text{for } v(u),$$

and from
$$u = \frac{C}{1+\alpha MC} \text{ to } +\infty \quad \text{and} \quad \frac{-C}{1-\alpha MC} \text{ to} -\infty \quad \text{for } w(u).$$

It therefore follows that the $\pm$ Fourier transforms have the regions of regularity $R^{\pm}$ shown in figure 1. In that diagram we have shown the branch cuts drawn with

$$\frac{C}{1+\alpha MC} > \frac{1}{1+M} \quad \text{and} \quad \frac{1}{1-M} > \frac{C}{1-\alpha MC}.$$

If this condition is not satisfied the order of the branch points on the real u-axis should be reversed.

Both half-range transforms can be seen to be analytic in the region of overlap between $R^{\pm}$, and the integration path in (2.9) is taken to lie in this strip, and specifically along the line $\arg u = \delta$.

The solutions to (2.10) and (2.11) are Bessel functions of order m. We require that the solution be finite at $r = 0$ and decay as $r \to \infty$ for u in the strip. Hence

$$\left.\begin{aligned} P(u,r) &= A(u)\, J_m(kvr) \qquad (r < a), \\ &= B(u)\, H_m^{(2)}(kwr) \quad (r > a). \end{aligned}\right\} \tag{2.13}$$

Defining, further, the half-range transforms of the boundary displacement as $Z^{\pm}$, the boundary conditions (2.3) and (2.4) become

$$P_j^+(u,a^-)+P_i^+(u,a^-) = P_0^+(u,a^+), \tag{2.14}$$

$$Z^-(u) = 0, \tag{2.15}$$

in which P_j, P_0 are the transforms of the pressure in $r < a$, $r > a$ respectively and P_i is the transform of the incident pressure,

$$P_i^+(u,a^-) = \int_0^\infty \frac{J_m(j'_{mn}a^-)}{J_m(j'_{mn}a)}\exp(-i\mu_{mn}x+ikux)\,dx = \frac{1}{i(\mu_{mn}-ku)}, \tag{2.16}$$

for u in R^+.

We solve (2.14) and (2.15) by noting that

$$P_j(u,a^-)-P_0(u,a^+)+P_i^+(u,a^-) = P_j^-(u,a^-)-P_0^-(u,a^-) = F^-, \tag{2.17}$$

a function regular in R^-.

Using (2.5), we find that

$$P_j(u,a^-) = \frac{Z(u)\,\rho_j c_j^2 k^2 D_j^2 J_m(kva)}{kvJ'_m(kva)}, \tag{2.18}$$

$$P_j(u,a^+) = \frac{Z(u)\,\rho_j c_j^2 k^2\gamma D_0^2 H_m^{(2)}(kwa)}{kwH_m^{(2)}(kwa)}, \tag{2.19}$$

where $D_j^2 = (1-Mu)^2$, $D_0^2 = (1-\alpha Mu)^2$. Whence, substituting (2.18), (2.19) and (2.16) in (2.14), and noting that, from (2.15), $Z^-(u) = 0$, we find that

$$K(u)\,Z^+(u)+\frac{1}{i(\mu_{mn}-ku)} = F^-(u), \tag{2.20}$$

where

$$K(u) = \rho_j c_j^2 k^2\left[\frac{D_j^2 J_m(kva)}{kvJ'_m(kva)}-\frac{\gamma D_0^2 H_m^{(2)}(kwa)}{kwH_m^{(2)\prime}(kwa)}\right]. \tag{2.21}$$

We solve (2.20) by the Wiener–Hopf technique, described for example in Noble (1958).

We factorize $K(u)$ as $K(u) = K^+(u)\,K^-(u)$, where $K^{\pm}(u)$ are analytic and non-zero in the two half-planes R^+ and R^-. Then (2.20) gives

$$Z^+(u)\,K^+(u)+\frac{1}{K^-(u)\,i(\mu-ku)} = \frac{F^-(u)}{K^-(u)}. \tag{2.22}$$

This may be rewritten as

$$Z^+(u)\,K^+(u)+\frac{1}{K^-(\mu/k)\,i(\mu-ku)} = \frac{F^-(u)}{K^-(u)}-\left(\frac{1}{K^-(u)}-\frac{1}{K^-(\mu/k)}\right)\frac{1}{i(\mu-ku)}. \tag{2.23}$$

The left- and right-hand sides of this equation are analytic in the respective half-planes $R^{\pm}$. By an extension of Liouville's theorem they must therefore both equal some function $\bar{C}(u)$, which is regular over the whole u-plane, except at infinity. The only form of $\bar{C}(u)$ that has the required properties is a polynomial in u.

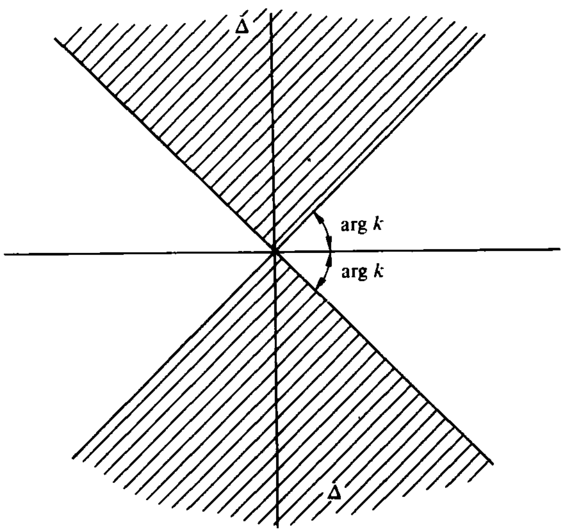

FIGURE 2. The complex k-plane showing the region Δ.

We consider now the edge conditions on η, and the resulting constraints on the behaviour of $Z(u)$. From Munt (1977), we find that if $k \in \Delta$, where Δ is a region of the k-plane determined by the instability zeros u_0, u_0^* of $K(u)$, then $K^+(u) = O(u^{\frac{3}{4}})$, $K^-(u) = O(u^{-\frac{1}{4}})$ as $|u| \to \infty$ but, if $k \notin \Delta$, then

$$K^+(u) = O(u^{\frac{1}{4}}), \quad K^-(u) = O(u^{\frac{1}{4}}) \quad \text{as} \quad |u| \to \infty.$$

The region Δ is shown in figure 2.

Then if $k \in \Delta$, we find that if $\eta(x)$ is $O(x^n)$ then $Z(u)$ is $O(u^{-(n+1)})$, and the left-hand side of (2.23) is $O(x^{-(n-\frac{1}{4})})$. This means that $\bar{C}(u)$ is a polynomial of order $\frac{1}{4} - n$. For instance if $n = \frac{3}{2}$ and the solution obeys the Kutta condition, we are restricted to $\bar{C}(u) = 0$. If the solution is the least singular one *not* obeying a Kutta condition, then $\bar{C}(u)$ must be a constant. The procedure for obtaining a causal solution in either case is to solve equation (2.23) for some $k \in \Delta$, for example with $\delta = \frac{1}{2}\pi$, and then argue the result for real k by analytic continuation.

Hence we find that

$$Z^+(u) = \left\{ \frac{-1}{i(\mu - ku)\, K^+(u)\, K^-(\mu/k)} + \frac{\bar{C}(u)}{K^+(u)} \right\} \tag{2.24}$$

as $Z^-(u) = 0$; this is also the value of $Z(u)$. Then by (2.9)

$$\eta(x, k) = \int_{-\infty \exp i\delta}^{+\infty \exp i\delta} [Z^+(u)]\, e^{-ikux}\, k\, du. \tag{2.25}$$

As δ passes from $\delta = \frac{1}{2}\pi$ to $\delta = 0$, the pole at $u = u_0$ is passed over by the integration path for u. For a causal solution we require, from a theorem of Jones & Morgan (1974), that $\eta(x, k)$ is analytic in the lower half-plane, and that

$$\exp\,[(ib + d)\, k]\, \eta(x, k) = O(|k|^p) \quad \text{as} \quad |k| \to \infty,$$

where b, d are real numbers, and $b > 0$. Therefore there must be no discontinuity in η as the contour passes over u_0. Therefore, we must add on a residue contribution from the pole for $\delta < \arg u_0$, and thus for a causal solution we have

$$\eta(x,k) = \frac{1}{2\pi}\int_{-\infty \exp i\delta}^{+\infty \exp i\delta} Z^+(u)\, e^{-ikux}\, k\, du + H(\arg u_0 - \delta) \lim_{u\to u_0} [iZ^+(u_0)(u-u_0)\,k]\, e^{-iku_0 x}. \tag{2.26}$$

With reference to Morgan (1974) it can be seen that the solution is causal whether $\bar{C}(u) = 0$ (Kutta condition) or $\bar{C}(u)$ is a constant (no Kutta condition).

Then the causal solution subject to a Kutta condition is given by

$$Z(u) = \frac{i}{K^+(u)\,K^-(\mu/k)\,(\mu - ku)}, \tag{2.27}$$

$$P_j(u) = \frac{i\rho_j c_j^2 k^2 D_j^2 J_m(kvr)}{kvJ_m'(kva)\,K^+(u)\,K^-(\mu/k)\,(\mu-ku)}, \tag{2.28}$$

$$P_0(u) = \frac{i\rho_j c_j^2 k^2 \gamma D_0^2 H_m^{(2)}(kwr)}{kwH_m^{(2)\prime}(kwa)\,K^+(u)\,K^-(\mu/k)\,(\mu-ku)}. \tag{2.29}$$

The general properties of the split functions have been given by Munt. We list them in the low-frequency limit in appendix A.

We consider first, and in most detail, solutions for an incident plane acoustic wave. In this limit the split functions are

$$K^+(u) = -\frac{2M^2a(u-u_0^*)(u-u_0)\rho_j c_j^2 k^2}{(ka)^2 v^+}, \quad K^-(u) = \frac{1}{v^-}. \tag{2.30}$$

The pressure perturbation, for $r < a$, is given by

$$p_j(x) = -\frac{1}{2\pi}\int_{-\infty \exp i\delta}^{\infty \exp i\delta} \frac{\rho_j c_j^2 k^2 D_j^2 J_m(kva)\, k e^{-ikux}\, du}{kvJ_m'(kva)\, i(\mu - ku)\, K^+(u)\, K^-(\mu/k)}. \tag{2.31}$$

In the limit of very low frequency

$$\left.\begin{aligned} K^+(u) &= \frac{-2a(1-Mu)^2\rho_j c_j^2 k^2}{(ka)^2(1-(1+M)u)}, \\ \frac{\mu}{k} &= \frac{1}{1+M}, \\ K^-(\mu/k) &= \tfrac{1}{2}(1+M), \end{aligned}\right\} \tag{2.32}$$

and therefore

$$p_j(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{e^{-ikux}\, du}{(1-M^2)\left(u - \dfrac{1}{1+M}\right)\left(u + \dfrac{1}{1-M}\right)}. \tag{2.33}$$

The value of this integral is equal to one or other of the pole residue contributions according as x is greater than or less than zero. The pole at $u = 1/(1+M)$ cancels out the incident field for $x > 0$, while the pole at $u = -1/(1-M)$ gives the reflected field inside the jet pipe. This has a value $p_j = -\exp[ikx/(1-M)]$, and therefore the reflection coefficient R for incident plane waves is, in the low-frequency limit,

$$R = -1. \tag{2.34}$$

The result (2.34) is the basis for Bechert's (1979) simple theory of nozzle-flow sound attenuation; it is entirely dependent on the satisfaction of a Kutta condition at the nozzle lip (see later).

It is clear that in this low-frequency limit there are no contributions to the pressure from instability waves. This follows from our approximation to $K^+(u)$ in which we set the instability poles u_0, u_0^* at $1/M$, rather than the more exact value (see appendix A) of $(1/M)(1 \pm i\sigma)$. If the latter value had been used in $K^+(u)$, there would have been contributions to the pressure from these two instability waves, growing and decaying exponentially along the jet. The value of the contribution to the pressure from these two poles is $O(\sigma^2)$, which is negligible for low-enough frequency. The axial velocity in the jet does, however, contain contributions corresponding to the instability waves, and it is given exactly by

$$u_x = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_j c_j k^2 D_j^2 J_m(kva)\, e^{-ikux}\, ku\, du}{kvJ_m'(kva)\, i(\mu - ku)\, K^+(u)\, K^-(\mu/k)\,(1-Mu)\,\rho_j c_j}, \tag{2.35}$$

which in the low-frequency limit becomes

$$u_x = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{u\, e^{-ikux}\, du}{\rho_j c_j (1-M^2)\left(u - \frac{1}{1+M}\right)\left(u + \frac{1}{1-M}\right)(1-Mu)}.$$

The contributions from the poles $u = 1/(1+M)$, $-1/(1-M)$, are the acoustic waves discussed above. The contribution from the pole at $u = 1/M$, gives the velocity fluctuation

$$u_x = \frac{2}{\rho_j c_j} \exp(-ikx/M). \tag{2.36}$$

We see that this represents a convected instability wave, albeit of vanishingly small growth rate. If we had used the more complete form of $K^+(u)$, u_x would have been given by

$$u_x = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{2u(1-Mu)\, e^{-ikux}\, du}{\rho_j c_j M^2 (1-M^2)\left(u - \frac{1}{1+M}\right)\left(u + \frac{1}{1-M}\right)\left(u - \frac{1-i\sigma}{M}\right)\left(u - \frac{1+i\sigma}{M}\right)}, \tag{2.37}$$

and the instability-wave contribution to this gives

$$u_x = \frac{e^{-ikx/M}}{\rho_j c_j}\left[e^{k\sigma x/M} + e^{-k\sigma x/M}\right], \tag{2.38}$$

which tends to the previous result (2.36) for $x \ll 1/\sigma$, but displays clearly the amplification and decay factors of spatial instability waves convected with the jet speed.

We can also derive the jet displacement due to these instability waves: it is

$$\eta(x) = \frac{a\, e^{-ikx/M}}{\rho_j c_j^2 M^2}\left[\frac{\sinh(kx\sigma/M)}{2\sigma/M}\right], \tag{2.39}$$

For small x, $x \lesssim M/k\sigma$, this expression shows that the shear-layer displacement grows linearly with distance from the end of the pipe. This expression is only valid for $x \gg a$, however, since our expression for $K^+(u)$ was only valid for $uka \ll 1$, and the behaviour near $x = 0$ depends on the value of $K^+(u)$ as $u \to \infty$. Despite this, however, the approximate form of $\eta(x)$ does at least give zero displacement at $x = 0$, even if the slope of the shear layer is non-zero; the exact equation for $\eta(x)$ does of course have zero slope at $x = 0$.

The linear growth of the amplitude of the instability waves with distance has a simple physical explanation. Consider a jet with an instability wave of negligible growth rate whose axial velocity fluctuation is $u_x = \hat{u}\exp[i\omega(t-x/U_j)]$. For this wave, the pressure fluctuation is zero. Therefore the continuity equation can be written

$$\frac{1}{r}\frac{\partial vr}{\partial r}+\frac{\partial u}{\partial x} = 0,$$

where v is the radial velocity, and hence

$$v = \frac{i\omega r}{2U_j}\hat{u}\exp[i\omega(t-x/U_j)].$$

Now the displacement of the jet boundary is related to the velocity v by

$$\frac{\partial \eta}{\partial t}+U_j\frac{\partial \eta}{\partial x} = v.$$

We assume that η is of the form $\eta = \hat{\eta}(x)\exp[i\omega(t-x/U_j)]$, and then it is clear that if $\eta(0) = 0$

$$\hat{\eta} = \frac{\hat{u}}{U_j}\frac{i\omega a}{2U_j}x. \tag{2.40}$$

This is precisely the relation (2.39) obtained from the exact analysis.

We consider next the sound field outside the jet. The pressure fluctuation is, from (2.29),

$$p_0 = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\rho_j c_j^2 k^2\gamma D_0^2 H_m^{(2)}(kwr)\,e^{-ikux}\,k\,du}{kwH_m^{(2)\prime}(kwa)\,(\mu-ku)\,K^+(u)\,K^-(\mu/k)}, \tag{2.41}$$

and in the far field this expression is best evaluated by the method of stationary phase. The stationary point is at

$$u = C\cos\theta/(1+\alpha MC\cos\theta), \tag{2.42}$$

so that

$$p_0 = \frac{e^{-ikc_jR/c_0}}{4\pi R(1+\alpha MC\cos\theta)}\left[\frac{4}{i}\frac{kwH_m^{(2)}(kwa)\,(\mu-ku)\,K^+(u)\,K^-(\mu/k)}{k\rho_j c_j^2 k^2\gamma D_0^2}\right], \tag{2.43}$$

evaluated at

$$u = \frac{C\cos\theta}{1+M\alpha C\cos\theta},$$

where (R,θ) is the position of the far-field observer in so-called 'emission-time' co-ordinates (figure 3). The bracketed term becomes, on substituting for $K^{\pm}$ from appendix A,

$$\frac{2ka\pi a\gamma}{(1-MC(1-\alpha)\cos\theta)^2}. \tag{2.44}$$

By substituting for γ, C, M we can then rewrite p_0 as

$$p_0 = \frac{A_j}{4\pi R}\frac{i\omega\rho_0(2p_i/\rho_j c_j)\exp[-i\omega R/c_0]}{(1+\alpha MC\cos\theta)(1-MC(1-\alpha)\cos\theta)^2}, \tag{2.45}$$

where A_j is the duct area, and $2p_i/\rho_j c_j$ may be recognized as the velocity fluctuation u_N at the exit of the duct. The significant features of this formula lie in the scaling

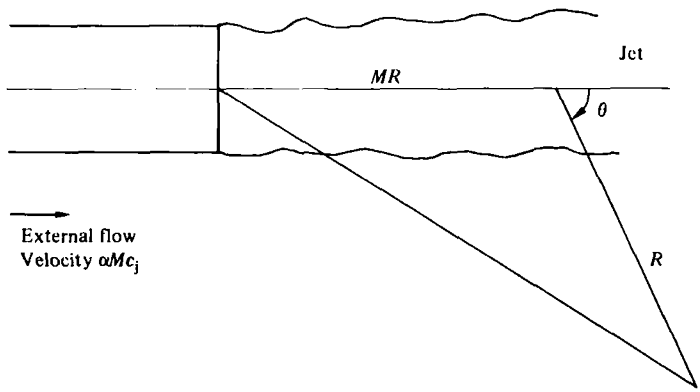

FIGURE 3. The emission-time co-ordinates R, θ.

of the level (for a given u_N) on the *far-field density* rather than the *jet density*, and in the field shape. The field shape is determined by the product of a Doppler factor based on the external flow velocity and the square of a Doppler factor based on relative flow velocity. This latter dependence is characteristic of low-frequency acoustic sources placed inside infinite jets (Goldstein 1975; Dowling *et al.* 1978). For angles close to 90° ($M\cos\theta \ll 1$) the effect of 'flight' (i.e. external flow) is represented by a factor $(1+\alpha MC\cos\theta)^{-6}$ on the intensity, which is identical with that found experimentally by Pinker & Bryce (1976).

We now discuss the flows of energy in this problem. In the jet pipe the net power flow is given by

$$W_N = \frac{A_j p_i^2}{\rho_j c_j}[(1+M)^2 - |R|^2(1-M)^2], \tag{2.46}$$

where p_i is the strength of the incident wave field, and R is the pressure reflection coefficient. In the limit of low frequency we have shown that $R = -1$, so that

$$W_N = 4M\frac{A_j p_i^2}{\rho_j c_j}. \tag{2.47}$$

This implies that there is a net flux of acoustic energy along the pipe, proportional to the Mach number, and independent, to first order, of frequency. This is in contrast wih the case of zero flow, where the net energy flux is of order

$$W_N = \frac{A_j p_i^2}{\rho_j c_j}(ka)^2.$$

In the jet the only significant motion is associated with the convected instability waves. Since there is negligible fluctuation in pressure associated with them, the only energy flow is a flux of kinetic energy. This is the product of the fluctuations in kinetic energy and mass flow, so that the net energy flux in the jet is

$$\begin{aligned} W_j &= \overline{(\rho_j u'_N A_j)(U_j u'_N)} \\ &= 4M\frac{A_j p_i^2}{\rho_j c_j}, \end{aligned} \tag{2.48}$$

where we have taken the velocity fluctuation as $u_N = 2p_i/\rho_j c_j$. It is clear that in this low-frequency limit there is a *total conversion of acoustic energy into kinetic energy associated with the instability waves*. We shall show later that this conversion of energy is critically dependent on the imposition of a Kutta condition.

The power radiated to the far-field is found to be (Munt 1982*b*)

$$W_R = \int_0^\pi 2\pi R^2 \frac{p^2(\theta, R)}{\rho_0 c_0} \sin\theta (1+\alpha MC\cos\theta)^2 \, d\theta, \tag{2.49}$$

which on substituting for the far-field pressure level $p(\theta, R)$ from (2.45), integrating and putting $A_j = \pi a^2$ gives

$$W_R = \left(\frac{p_i}{\rho_j c_j}\right)^2 \frac{\omega^2 \rho_0 a^2 (1+\frac{1}{3}M_R^2)}{c_0(1-M_R^2)^3}, \tag{2.50}$$

where $$M_R = MC(1-\alpha) = (U_j - U_0)/c_0.$$

This expression is the product of the net incident energy in the pipe in the absence of flow, the square of the compactness ratio ka of the jet, the ratio of the impedances of the jet and ambient medium, and a factor which depends on the velocity of the jet relative to the surrounding fluid and which causes the power radiated to increase rapidly as $M_R \to 1$. When the jet and ambient fluid have the same velocity the power radiated is unaffected by Mach number. The singularity in the radiated power when $M_R = 1$ could have been avoided by using a more accurate expression for $K^+(u)$; then the Doppler factors $1 - MC(1-\alpha)\cos\theta$ would have been replaced by

$$[(1-MC(1-\alpha)\cos\theta)^2 + \sigma^2 M^2]^{\frac{1}{2}}$$

and the singularity at the Mach angle removed.

In their study of jet noise, Dowling *et al.* (1978) have shown that, when a low-frequency acoustic source is placed in a jet, the radiation from it changes dramatically if the temperature of the jet increases to such an extent that 'it is hotter than it is compact', that is when $\rho_0/\rho_j \gg (ka)^2 \ln(ka)$. We analyse the propagation of sound out of a jet pipe in this limit.

The sound pressure outside the jet is again given by (2.41). When the jet is very hot we have shown (appendix A) that the form of the split functions changes, and on substituting for the split functions in the low-frequency limit the far-field pressure is now given by

$$p_0 = \frac{A_j i\omega\rho_j u_N (1+M)\, e^{-i\omega R/c_0}}{2\pi^2 R(1+M\alpha C\cos\theta)^2 (ka)^2 \ln(kaC)\cos\theta}. \tag{2.51}$$

Compared with the previous result, the field shape for the light jet does not show the downstream Doppler amplification, but is infinite in the side-line (90°) direction. We see therefore that the 'light-jet' condition always fails at this position, as one would expect from the fact that that condition is a compactness condition, which is necessarily violated around the Mach angle, which in the high-temperature case is the 90° position. The field in the jet is given by (2.31) and, substituting for K^+, K^-, we find the pole contributions at $u = 1/(1+M)$, $-1/(1-M)$, $-i\epsilon$. The first of these does not, unlike the case considered earlier, cancel the incident field, which now continues to propagate along the jet. In this limit the jet itself behaves, to first order, as a semi-infinite rigid tube. The pressure due to this pole is given by

$$p_j \sim \frac{1}{\pi}\epsilon^2(1+M)\,e^{-ikx/(1+M)}, \tag{2.52}$$

and the level in the jet due to the pole at $u = i\epsilon$ is given by

$$p_j \sim \frac{2}{\pi}\epsilon^2\,e^{-\epsilon kx}\,(1+M). \tag{2.53}$$

Both these fields (which arise for $x > 0$) are small in the light-jet limit (ϵ small).

The reflected sound field in the jet pipe is given by the contribution from the pole at $u = -1/(1-M)$, namely

$$p_j = \frac{(1-M^2)\,\epsilon^3}{\pi}\,e^{ikx/(1-M)}. \tag{2.54}$$

Clearly then, the reflection coefficient is of order $[\gamma(ka)^2\ln(ka)]^{-1}$, which is small in this light-jet limit.

We have thus shown that if the jet is sufficiently hot then there is a radical change in the acoustic behaviour of the jet-pipe system. Most of the sound is no longer reflected back up the pipe, but continues trapped inside the jet. The reason for this is seen by examining the relation between the pressure gradient and displacement on the jet boundary:

$$\frac{D^2\eta}{Dt^2}+\frac{1}{\rho_0}\frac{\partial p}{\partial r} = 0.$$

From this it is clear that if ρ_0 is greatly increased, and tends to infinity, then for a given value of pressure gradient the boundary displacement must tend to zero. In the far field the radiation is reduced compared with that for the non-light-jet case, except for angles close to the side-line direction. At this 90° position (corresponding to the Mach angle), the compactness condition of the light jet does not hold, as already observed. An addition feature of the light-jet limit is that the instability waves on the jet column are suppressed.

We now consider, in less detail, the radiation from higher-order spiral duct modes. In the low-frequency limit it is well known that in the absence of a mean flow sound radiates very inefficiently. Substituting for the split functions $K^{\pm}(u)$ from (A 14) in the expression for the sound field outside the jet, we find that, in the low-frequency limit, the radiation field is given by

$$p_0 = \frac{a\,e^{-i\omega R/c_0}(ka)^m\sin^m\theta(8\pi/m)\,\gamma}{4\pi R(1+\alpha MC\cos\theta)^m\,[(\mu/k)\,(1+\alpha MC\cos\theta)-\cos\theta]\,[(1-MC(1-\alpha)\cos\theta)^2+\gamma]}. \tag{2.55}$$

This formula shows that, as the mode number m increases, the power radiated at a given (low) frequency progressively decreases. The radiation is predominantly in the side-line direction, and the effect of flight is largely that of the Doppler factors $1+\alpha MC\cos\theta$ which shift the field further forwards for higher values of α. For sufficiently low frequencies, μ/k becomes $ij'_{mn}/ka(1-M^2)^{\frac{1}{2}}$, which is much greater than unity, so that the far-field pressure can be written

$$p_0 = \frac{a\,e^{-i\omega R/c_0}(ka)^{m+1}\,8\pi(1-M^2)^{\frac{1}{2}}\sin^m\theta}{4\pi R(1+\alpha MC\cos\theta)^{m+1}\,mij'_{mn}}\,\frac{\gamma}{(1-MC(1-\alpha)\cos\theta)^2+\gamma}. \tag{2.56}$$

The radiation from all of these higher-order modes varies as a higher power of ka than that from the plane-wave mode. Further, for a *given* pressure level at infinity upstream, the pressure level at the nozzle, on which this radiation scales, is exponentially small.

The field in the jet and pipe, $r < a$, is obtained as before. In the usual limit of both low Helmholtz number and low Strouhal number, the pressure field is given by

$$p_j = \left(\frac{r}{a}\right)^m \frac{\exp\left[-ik(1+i\gamma)x/(1+i\alpha\gamma)M\right](1-M^2)^{\frac{1}{2}}ka}{2j'_{mn}M(1+\alpha\gamma^2)}. \tag{2.57}$$

Unlike the plane-wave case considered earlier, the instability wave *does* now have a pressure disturbance associated with it, which increases in proportion to the Strouhal number for a given initial amplitude. The growth rate is given by $k\gamma(1-\alpha)/M(1+\alpha^2\gamma^2)$, which vanishes when there is no velocity difference across the jet boundaries. Further, these non-axisymmetric waves are amplified with distance downstream, even for the lowest frequencies, although there the initial disturbance level is very small, owing to the aforementioned dependence on frequency, and the exponentially small level of the sound at the pipe exit due to the spiral acoustic modes being cut off in the pipe.

2.2. *Subsonic jet with no Kutta condition*

We describe here two types of problem relating to a jet with no Kutta condition. Firstly we consider the case where a jet eigensolution is added on to the Kutta-condition solution, by taking $\bar{C}(u)$ in (2.24) as constant. Secondly we adopt an approach due to Howe (1979). Instead of assuming the existence of an instability wave in the jet, he assumes that the jet motion consists of some other *neutrally stable* wave which is convected at a Mach number, νM, less than the jet Mach number M. He then finds the field due to this and matches it to the nozzle flow. He adds this field to that found with no Kutta condition, and forces the total to satisfy a Kutta condition.

We examine first the case where $\bar{C}(u)$ is a constant C_0, say. Then (2.27)–(2.29) become

$$Z(u) = \frac{1}{K^+(u)}[E+C_0], \tag{2.58}$$

$$P_j(u) = \frac{-\rho_j c_j^2 k^2 D_j^2 J_m(kvr)}{kvJ'_m(kva)K^+(u)}[E+C_0], \tag{2.59}$$

$$P_0(u) = \frac{-\rho_j c_j^2 k^2 D_0^2 H_m^{(2)}(kwr)}{kwH_m^{(2)\prime}(kwa)K^+(u)}[E+C_0], \tag{2.60}$$

where we have defined
$$E = \frac{-1}{i(\mu-ku)K^-(\mu/k)}.$$

By choosing different (non-zero) values of C_0, we can obtain a whole range of solutions, none of which obeys a Kutta condition. One of these is of special interest: that where C_0 is chosen to remove the instability pole u_0. We choose

$$C_0 = \frac{1}{K^-(\mu/k)\,i(\mu-ku_0)}, \tag{2.61}$$

and then
$$E+C_0 = \frac{-k(u-u_0)}{ikK^+(\mu/k_0)(\mu-ku_0)(\mu-ku)}. \tag{2.62}$$

Clearly, the net effect of this is to multiply the Kutta condition solutions in u by $(u-u_0)/(\mu/k-u_0)$. When the constant C_0 is chosen in this way, the formula for the far-field sound (2.45) then becomes

$$p_0 = \frac{\rho_0 u_N}{4\pi R} i\omega \frac{A_1(1+M)\,e^{-i\omega R/c_0}}{(1+\alpha MC\cos\theta)^2\,(1-MC(1-\alpha)\cos\theta)}. \tag{2.63}$$

The major difference between this and the previous result is the removal of one of the *relative*-velocity-based Doppler factors, which results in a considerably less directional sound field.

The corresponding multiplier for the field inside the pipe is obtained using

$$u = -1/(1-M),$$

so that the reflection coefficient has changed from -1 to $-(1+M)/(1-M)$. It is of particular interest that the power flow in the duct, which is proportional to

$$(1+M)^2 - |R|^2(1-M)^2,$$

is now *precisely zero*. Therefore, we conclude that the *imposition of a Kutta condition is essential for a transfer of power from acoustic to hydrodynamic fields to take place.* With no Kutta condition, and no generation of growing instability waves, almost all the incident energy is reflected back up the duct, a negligible $O(k^2a^2)$ fraction being diffracted to the far field.

Howe proceeds by adding on to this non-Kutta-condition solution the field due to a jet motion convected at a speed νMc_j. We assume that in the absence of the pipe this wave would induce a jet displacement $Z_1 e^{-ikx/\nu M}$.

Using this as the incident disturbance, we solve for the field in the same way as for the incident pressure wave. Then the value of Z is chosen so as to cancel the resulting singularity in velocity at the edge; so that the original solution is multiplied by the factor $(u-u_0)(1-M\nu\mu/k)/(\mu/k-u_0)(1-M\nu u_0)$. In the far field this must be evaluated at the stationary-phase value of u, to give, with $\mu/k = 1/(1+M)$, $u_0 = 1/M$,

$$(1+M(1-\nu))\frac{1-MC(1-\alpha)\cos\theta}{1-MC(\nu-\alpha)\cos\theta}.$$

Thus we see that as ν is varied between 0 and 1 the solution changes continuously from the non-Kutta-condition solution to the Kutta-condition solution. The major change in the far field, as compared with the Kutta-condition solution, lies in the replacement of the $1-MC(1-\alpha)\cos\theta$ Doppler factor by one based on the convection speed of the waves. This results in a less directional radiation field, with a corresponding reduction in the acoustic power radiated.

The reflection coefficient is obtained by substituting $u = -1/(1-M)$, giving a multiplier $(1+(1-\nu)M)/(1-(1-\nu)M)$. The reflection coefficient varies again from its Kutta- to non-Kutta-condition values as ν is altered. This is only to be expected, since with $\nu = 1$ the convected waves are indistinguishable from the instability waves, while with $\nu = 0$ there is no spatial variation, and the wave is effectively absent.

In this derivation of the radiation the existence of the waves convected at speed νMc_j is only an assumption. There are grave doubts over its validity, since the wave is not in fact the solution of any equations governing the motion of the fluid. Nevertheless, the idea remains a plausible means of representing in some way the characteristics of the real jet flow.

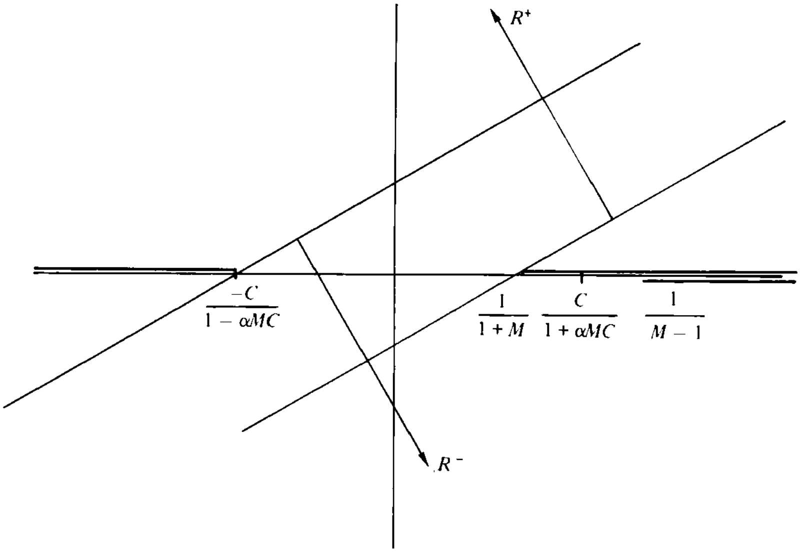

FIGURE 4. Positions of branch cuts and regions of regularity R^+, R^-, in the complex u-plane for supersonic flow.

2.3. *Supersonic jet*

When we come to the supersonic jet the basic equations are the same; the difference in the solution concerns only the position of the branch cuts of the u-plane. When $M \to 1$, the branch point at $-1/(1-M)$ goes to $-\infty$ for subsonic flow, but when $M > 1$ it reappears on the other side of the diagram at $1/(M-1)$. This is a consequence of the impossibility of waves propagating upstream against the supersonic flow. The resultant branch cuts and positions of R^+, R^- for $M > 1$ are essentially as described by Morgan (1974), and are shown in figure 4. It is assumed in this diagram that $1/(M-1) > C/(1-\alpha M) > 1/(M+1)$. If this is not so, the order of these points on the real axis is correspondingly changed.

We consider the case of an incident plane wave propagating down the jet pipe towards the exit,

$$p_i = \exp[-i\mu x], \quad \mu = k/(1+M), \tag{2.64}$$

and confine the analysis to plane waves purely for simplicity. There is no other reason for doing so here since all modes are cut on in supersonic flow. The derivation of the field due to the higher-order modes follows in an altogether similar fashion.

As $ka \to 0$, it is shown in appendix B that (unless $u \gg 1/ka$, $\alpha \neq 0$) $K^-(u) = 1$, $K^+(u) = K(u)$. Then the formulae used previously may be applied, with the previous value of $K^+(u)$ multiplied by $J_0(kva)/v$, and the factor $K^-(\mu/k)$ multiplied by $v^-(\mu/k)$. Consequently, all the Fourier-transformed quantities $P_j(u)$, $P_0(u)$, $Z(u)$ are multiplied by

$$\frac{v^-(u)}{v^-(\mu/k)} \frac{1}{J_0(kva)} = Q \quad \text{(say)}. \tag{2.65}$$

In the above discussion we did not mention the edge conditions to be satisfied near $x = 0$, and on which there was previously so much emphasis. At high kva ($u \to \infty$) the kernel $K(u)$ has a behaviour similar to its two-dimensional vortex-sheet equivalent.

The latter case has been examined in detail by Morgan, who finds that $K^+(u)$ is $O(u)$; $u \to \infty$, resulting in a displacement $\eta(x) \sim x$; $x \to 0^+$. The displacement is therefore continuous across $x = 0$, but its slope is not, so that a Kutta condition in the subsonic sense cannot be applied. However, one would not expect it to hold for this *unsteady* supersonic flow, any more than it does in *steady* supersonic flows, and it cannot because of the impossibility of the downstream motion of the jet affecting the edge. Besides that described above, further solutions corresponding to the subsonic non-Kutta-condition solutions could be obtained. These would be even more singular at the edge, and are physically implausible (displacement at the edge must at least be discontinuous).

We determine next the far-field sound level. This is given by its subsonic value with a Kutta condition, multiplied by the above factor (2.65), evaluated at

$$u = C\cos\theta/(1+\alpha MC\cos\theta).$$

The far-field pressure is accordingly

$$p_0 = \frac{i\omega A_j u_N \rho_0 e^{-i\omega R/c_0}}{4\pi R}\frac{(1+M)[1-C\cos\theta(M(1-\alpha)-1)]}{(1-(1-\alpha)MC\cos\theta)^2(1+\alpha MC\cos\theta)^2}. \tag{2.66}$$

The interesting features of this formula are the large values of forward arc amplification (the exponent of $1+\alpha MC\cos\theta$ is increased) and the factor

$$1 - C\cos\theta(M(1-\alpha)-1).$$

The latter causes the field to have a zero if $C(M(1-\alpha)-1) > 1$.

The reflected field inside the pipe is now precisely zero, since the pole at

$$u = -1/(1-M)$$

is no longer present. In fact the field inside the pipe is precisely zero everywhere, since all the poles representing cut-off waves inside the pipe are now in R^-, and cannot contribute for $x < 0$.

We consider the field in the jet in more detail; it is given by

$$p_j = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{J_0(kvr)\,e^{ikux}\,du}{J_0(kva)\left(\dfrac{1}{1+M}-u\right)}. \tag{2.67}$$

The pole at $u = 1/(1+M)$ cancels the incident field. The other poles at $J_0(kva) = 0$ are the unsteady flow analogue of the steady wave structure of an imperfectlyexpanded supersonic jet. (We did not consider them for the subsonic jet, since there they represented fields that decayed exponentially along the jet axis.) These poles occur at $kva = j_n'$. Substituting in the formula for the pressure inside the jet and ignoring terms of $O(ka)$, we find that p_j can be expressed as the sum of contributions from these poles:

$$p_j = \sum_{n=1}^{\infty}\frac{-2J_0(j_n r/a)}{j_n J_0(j_n)}\cos\frac{j_n x}{\beta a}. \tag{2.68}$$

This formula, which is composed of contributions from the quasi-periodic wave structure, is valid for $x \ll 1/ka$. A form valid over the whole distance x is only obtained by use of more exact approximations for the poles. In addition to these contributions to the pressure in the jet, there is again an instability wave present, which only affects the velocity (see § 2.1).

We next consider the energy flows involved. Inside the duct the energy flux is that in the incident sound wave alone, and is accordingly

$$W_0 = \frac{p_i^2 A_j}{\rho_j c_j}(1+M)^2. \tag{2.69}$$

The energy in the instability wave is

$$W_i = (\rho A u')(u'U) = \frac{(1+M)^2}{M}\frac{p_i^2 A}{\rho_j c_j}. \tag{2.70}$$

Clearly the energy in this wave is not equal to the net energy in the jet pipe. However, there are now two additional contributions to the energy – that due to the coherent wave structure alone, and that due to the interference field between it and the instability wave. This may be contrasted with the work of Howe & Ffowcs Williams (1978), who considered the scattering of the coherent wave by random shear-layer turbulence. They found that all of its energy was scattered into sound, there being no coupling of the coherent structure and instability waves.

The energy flux across a section of the jet is given by $W = \int dA\, h_0'(\rho v_n')$, where h_0' is stagnation enthalpy. Splitting this into components due to the coherent wave structure, p_c, u_c, and the instability wave u_i ($p_i = 0$), we obtain

$$W = \int \left(\frac{p_c}{\rho} + Uu_c + Uu_i\right)\left(\frac{Up_c}{c^2} + \rho u_c + \rho u_i\right) dA. \tag{2.71}$$

Now the fluctuation due to the coherent wave structure is to first order quasi-static, so that $p_c/\rho + u_c U = 0$. Therefore, the contribution to the integral due to this wave structure alone is negligible, and the only important term (apart from the instability term already calculated in (2.70)) is the interference term

$$W_{\text{int}} = \int u_i U(p_c U/c^2 + u_c \rho)\, dA. \tag{2.72}$$

Substituting for p_c, u_c, u_i, and integrating over the area of the jet,

$$W_{\text{int}} = 4\frac{M^2-1}{Mc_j}\sum_{-\infty}^{+\infty}\frac{\cos(j_n x/\beta a)}{j_n^2}\, e^{-i\omega x/U_j}\frac{p_i^2 A_j}{\rho_j c_j}. \tag{2.73}$$

For finite x, the cosine terms dominate, and the exponential can be ignored.

We now consider the flux of energy through the walls of the jet. This is given by $W_N = \int dS\, dx\,(Uu_i)\, v_c\, \rho$, where v_c is the radial velocity in the coherent field, and x is the length of the jet, perimeter S. The velocity in the coherent field is given by

$$U\,\partial v_c/\partial x = -\rho^{-1}\,\partial p/\partial r,$$

so that using (2.68)

$$v_c = \frac{\beta}{M}\sum_{-\infty}^{+\infty}\frac{2\sin(xj_n/\beta a)\,J_0'(j_n r/a)}{j_n J_0'(j_n)}\frac{p_i}{\rho_j c_j}. \tag{2.74}$$

Hence, substituting in the expression for W_N and integrating from 0 to x,

$$W_N = \sum_{-\infty}^{+\infty} 4\pi a\,\frac{(M^2-1)(M+1)}{Mj_n^2}[\cos(j_n x/\beta a) - 1]\frac{p_i^2}{\rho_j c_j}. \tag{2.75}$$

This contribution, when added to the previous one, gives the total power through the walls and cross-section of the jet up to a certain distance x as

$$W_J = \sum_1^\infty \frac{4A_J p_i^2}{\rho_J c_J} \frac{(M^2-1)(M+1)}{M_J j_n^2} = \frac{A_J p_i^2}{\rho_J c_J} \frac{(M^2-1)(1+M)}{M}. \tag{2.76}$$

Adding W_J to the net power W_i, in the jet instability wave, we obtain a total power of

$$\frac{Ap_i^2}{\rho_J c_J} \frac{(1+M)^2}{M}((M-1)+1) = \frac{A_J p_i^2}{\rho_J c_J}(1+M)^2. \tag{2.77}$$

This is precisely equal to the power in the incident wave. We have therefore shown that, with a *supersonic* jet, all the power in the incident jet-pipe acoustic wave is converted into either hydrodynamic kinetic energy or into the power in the interference field between the quasi-steady wave structure and the instability wave. Indeed, in this case all the incident acoustic energy is in some sense absorbed, and the basic phenomenon found by Bechert *et al.* (1977) for a subsonic jet applies equally for a supersonic jet.

3. Scattering of an external sound field by a cylindrical pipe with flow

The present problem has also been solved approximately by Jacques (1975). He, however, finds a formula for the far-field scattered sound that is different from ours. We shall show in § 5 that his result is incorrect because in his application of the acoustic analogy he omits certain source terms.

3.1. *Subsonic jet*

We consider an incident plane acoustic wave with pressure $e^{-iku_1x-kv_1y}$, in which, if the wave vector is at an angle θ_0 to the jet axis, $u_1 = C\cos\theta_0/(1+\alpha MC\cos\theta_0)$.

We first split this up into its circumferential modes to find that the incident pressure field is

$$p_1 = \sum_0^\infty \epsilon_m(-i)^m J_m(kv_1 r)\cos m\phi\, e^{-iku_1 x}, \tag{3.1}$$

where $v_1 = v(\theta_0)$, $\epsilon_m = 1$ $(m=0)$, $\epsilon_m = 2$ $(m \neq 0)$.

To apply the theory of the § 2 we require the pressure that would have existed on the wall of the pipe had the pipe been infinite. To find this, we add on to each modal term an extra term $A_m H_m^{(2)}(kv_1 r)$ and apply $\partial\phi/\partial r = 0$ on $r = a^+$, to obtain

$$\epsilon_m(-i)^m = -A_m H_m^{(2)\prime}(kv_1 a), \tag{3.2}$$

so that

$$p = \sum_0^\infty \epsilon_m(-i)^m \frac{[H_m^{(2)\prime}(kv_1 a)J_m(kv_1 r) - J_m'(kv_1 a)H_m^{(2)}(kv_1 r)]\,e^{-iku_1x}}{H_m^{(2)\prime}(kva)}, \tag{3.3}$$

To the lowest order in ka we find that the plane wave $(m=0)$ component is $p = e^{-iku_1 x}$, while the first spiral mode component is

$$p = -4ikv_1 a\, e^{-iku_1x} \quad \text{on} \quad r = a. \tag{3.4}$$

The component (3.4), and the other components with $m > 0$ are smaller by a factor at least $(ka)^m$ than the $m = 0$ component as $ka \to 0$, and can accordingly be neglected.

We therefore concentrate on the plane-wave component only. For this external forcing, the derivation of the Wiener–Hopf equation proceeds in a similar manner to that of § 2. The only difference lies in the wavenumber of the incident field, which we take as u_1. Consequently, examining the rest of the theory, we see that the Fourier transformed pressures and displacements have their previous values (2.27)–(2.29) multiplied by the factor

$$-\frac{u-\mu/k}{u-u_1}\frac{K^-(\mu/k)}{K^-(u_1)}. \tag{3.5}$$

In the far field this must be evaluated at $u = C\cos\theta/(1-\alpha MC\cos\theta)$, and it then becomes

$$\frac{[1-C(1+M(1-\alpha)\cos\theta)][1+C\cos\theta_0(1-M(1-\alpha))]}{2(\cos\theta-\cos\theta_0)C}. \tag{3.6}$$

Compared with the field shape of internal noise this has two interesting features. Firstly, it is singular at $\theta = \theta_0$. This singularity is spurious, and is similar to that found in half-plane diffraction problems on geometrical-optics boundaries. It can be removed using an improved evaluation of the stationary-phase integral, taking account of the fact that when $\theta \simeq \theta_0$, the integrand has a pole near the stationary phase point. Secondly, the field is zero at the angle $\theta = \arccos[C/(1+(1-\alpha)M)]$. This is the 'cone-of-silence' angle for sound waves passing from the jet to the far field.

In addition to the scattered field discussed above there is an additional field present due to the pole at $u = u_1$. This cancels, in a manner entirely familiar in diffraction theory, the portion of the incident field that represented sound reflected off the duct walls, but it only exists for angles less than θ_0 to the jet axis.

To obtain the fields in the jet and pipe we again use the previous solution, multiplied by (3.5). The pole at u_1 represents the sound waves inside the jet due to the incident field. These are pressure waves of amplitude equal to that of the incident field, i.e. $p_j = p_i e^{-iu_1kx}$. The field reflected up the pipe is given by the pole at $u = -1/(1-M)$, for which the above multiplier is equal to -1. Therefore the amplitude of the reflected wave is equal to that of the incident wave. The pole at $u-1/M$ once again gives the instability waves, whose effects are felt only as an axial velocity surging, the pressure perturbation being absent. Then the above multiplier (3.5) is

$$-\frac{1}{2}\left(1-\frac{u_1}{1-Mu_1}\right), \tag{3.7}$$

so that the instability wave has an axial velocity fluctuation

$$u_x = -\frac{p_i}{\rho_j c_j}\left(1-\frac{u_1}{1-Mu_1}\right). \tag{3.8}$$

This completes our evaluation of the sound scattered when low-frequency plane waves are incident upon a cylindrical pipe with internal and external flows. It is of interest to compare our results with those of Jacques (1975). In his paper, he first derives the 'zeroth-order' fields in the jet and the pipe, neglecting the secondary sound radiation. Then he applies the acoustic analogy to determine the latter. It is clear that the zeroth-order fields which we derive here are identical with his approximate solutions. Our result differs only in the field shape of the radiated sound field, which is more complicated than his. The two results are unequal even in the low-Mach-number limit. We pursue the application of the acoustic analogy to this problem in some detail in §4.

In discussing the relevance of his model, Jacques supposes that the incident sound waves are caused by some near-field turbulent pressure fluctuation from, say, a nearby jet. Therefore he takes the incident pressure to scale as $p_i \sim \rho U^2$, where U is some turbulence velocity. Inserting that into either our or his formulae for the far-field sound gives sound levels scaling as $p \sim \rho U^2(U/c_0)(a/R)$, where it is assumed that the incident pressure has frequencies proportional to this velocity U. We feel his modelling to be inappropriate. If the pressure fluctuations do scale in this way, and are further the result of some nearby aerodynamic disturbance, then the incident sound field cannot be modelled as plane waves. In that case, a more appropriate modelling would involve quadrupole sources, as mentioned briefly in §6. In spite of our misgivings about Jacques' problem, it appears, none the less, that the scaling laws given by Jacques for aerodynamic disturbances are indeed correct.

In our treatment of the scattering of external sound waves we have neglected such factors as the finite growth rate of the instability waves, and the light-jet issue. In this problem, though, the basic phenomena they represent are no different from those with incident internal noise, and the corresponding results could easily be derived.

We consider briefly the effect of relaxing the Kutta condition at the exit of the pipe. Generally the changes, compared with the case where a Kutta condition does apply, are similar to those for internal noise. As before, we can relax the Kutta condition by choosing some constant value of $\bar{C}(u)$ in the Wiener–Hopf equation. When we choose $\bar{C}(u)$ so as to extinguish all the unstable waves in the jet, the overall effect is to multiply all the $P(u), Z(u)$ by $(u-u_0)/(u_1-u_0)$, where u_0 is the instability pole. For the far field this factor, with $u = C\cos\theta/(1+\alpha MC\cos\theta)$, becomes

$$\frac{(1+M\alpha C\cos\theta_0)(1+MC(1-\alpha)\cos\theta)}{(1+M\alpha C\cos\theta)(1+MC(1-\alpha)\cos\theta_0)}. \tag{3.9}$$

The effect on the field shape is to replace one of the relative-jet-velocity-based Doppler factors by one based on the relative velocity and incidence angle θ_0. There is increased Doppler amplification in the upstream arc due to external flow.

The field transmitted up the pipe is given by the above factor with $u = -1/(1-M)$, that is

$$\frac{1+\alpha MC\cos\theta_0}{(1-M)(1-M(1-\alpha)C\cos\theta_0)}. \tag{3.10}$$

This field is usually larger than that for the Kutta-condition case, and may become very large as $M \to 1$. The field in the jet arises now only from the pole at $u = 1/(1+M)$, since the pole at $1/M$ representing the instability wave is cancelled; with $u = 1/(1+M)$ we have to multiply the Kutta-condition solution by the factor

$$\frac{1+\alpha MC\cos\theta_0}{(1+M)(1-M(1-\alpha)C\cos\theta_0)}. \tag{3.11}$$

In the above discussion we have not considered the energy flows involved, as we did for incident internal noise. The issue is felt to be unimportant here, since there is no clear 'incident' energy flow to act as a reference point. The only useful reference quantity is the net acoustic energy flow inside the jet, due directly to the incident waves. Then there is an interesting counterpart to the acoustic energy conversion discussed earlier, in that some energy is converted to kinetic energy, which is carried away by the jet instability waves.

3.2. *Supersonic jet*

One of the most interesting aspects of Jacques' work is the prediction that the sound scattered vanishes when the jet is sonic. We have shown in § 3.1 that in our solution this does not occur. As we will see in § 4, fields such as those leading to the radiation field can be built up from monopoles and dipoles at the exit plane, dipoles on the outside wall of the pipe and quadrupoles on the downstream shear layers. Jacques considers just the first of these and finds them to vary as $1-M$. This neglects the quadrupole sources which do not vanish at $M = 1$.

We now examine the supersonic jet problem ($M > 1$). Then, with the same incident wave as in § 3.1, and with the same modifications to the internal-noise theory, we can use the theory of § 2.3. Then the functions $P(u), Z(u)$ are multiplied by the factor

$$-\frac{(u-\mu/k)\,K^-(\mu/k)}{(u-u_1)\,K^-(u_1)}, \tag{3.12}$$

which is equal to $-(u-\mu/k)/(u-u_1)$, since $K^- = 1$ for supersonic flows at low frequency.

The far-field sound level is therefore, for a given p_i, multiplied by this factor evaluated at $u = C\cos\theta/(1+\alpha MC\cos\theta)$, giving a multiplier

$$-\frac{[1-(1+(1-\alpha)\,MC\cos\theta)](1+\alpha MC\cos\theta_0)}{C(\cos\theta-\cos\theta_0)\,(1+M)}. \tag{3.13}$$

We notice that these factors are similar to those in the subsonic case, giving a sound field zero at the cone-of-silence angle $\theta = \arccos 1/(1+(1-\alpha)\,M)\,C$, and with the stationary-phase calculation failing at $\theta = \theta_0$. There is no hint of the field becoming zero when the Mach number approaches one.

The field in the pipe is still zero, as it was for internal noise, since the factor $-(u-\mu/k)/(u-u_1)$ is still finite with $u = -1/(1-M)$. The pressure fields in the jet are also multiplied by this factor. For the cellular wave structure, with $u \sim O(1/ka)$, the pressure amplitude is changed only in sign. The value of the instability-wave axial velocity is multiplied by the factor with $u = 1/M$, namely

$$-\frac{1+M\alpha C\cos\theta_0}{(1+M)\,(1-(1-\alpha)\,C\cos\theta_0)}. \tag{3.14}$$

In all this, the field representing the incident wave is, of course, $p = p_i e^{-iku_1 x}$, as it was for the subsonic case.

4. Acoustic analogies

In this section we use two forms of acoustic analogy to derive equations for the sound field. These enable the far-field sound to be ascribed to various monopole, dipole and quadrupole sources. The results are of interest for several reasons. In the past these analogies have been used alone to determine the far-field sound. In most cases this has been done incorrectly, ignoring the quadrupole sources. We show that, at high Mach numbers, these quadrupole sources are responsible for most of the far-field sound. Further, we show how the $O(1)$ fields induced in the pipe and jet may be deduced by simple reasoning in the low-frequency limit, without reference to the

Wiener–Hopf solution to the complete problem. We then use these zeroth-order fields to evaluate the source terms.

We consider two forms of the acoustic analogy: that derived from Lighthill's (1952) equations, and a different analogy, due to Dowling *et al.* (1978), incorporating a mean flow. An alternative analogy is that of Howe (1975), which relates the sound field to unsteady vorticity. Howe (1979) has used it to discuss the transmission of sound out of a pipe with flow with results similar to those of our analysis, but restricted to low Mach numbers.

4.1. *The Lighthill analogy*

Ffowcs Williams & Hawkings (1969) have shown that the equation governing the sound field created by a moving surface defined by $f(\mathbf{x}) = 0$ and moving at a speed v is

$$\left(\frac{\partial^2}{\partial t^2}-c_0^2\frac{\partial^2}{\partial x_i^2}\right)H(f)(\rho-\rho_0)=\frac{\partial^2 T_{ij}H(f)}{\partial^2 x_i\,\partial x_j}-\frac{\partial}{\partial x_i}\left[(p_{ij}+\rho u_i(u_j-v_j))\,\delta(f)\frac{\partial f}{\partial x_j}\right]$$
$$+\frac{\partial}{\partial t}\left[(\rho_0 v_i+\rho(u_i-v_i))\,\delta(f)\frac{\partial f}{\partial x_i}\right], \quad (4.1)$$

where $\mathbf{u}$ is the fluid velocity, $T_{ij} = \rho u_i u_j + p_{ij} - c_0^2(\rho-\rho_0)\,\delta_{ij}$ is the Lighthill acoustic stress, and p_{ij} the compressive stress tensor. We apply this to a surface that encloses the end of the nozzle and the outer walls of the pipe. To take account of the external flow, we express the solution to this equation in convected co-ordinates such that the nozzle is fixed relative to the observer. Then, the Green function G satisfying

$$\left[\frac{D^2}{Dt^2}-c_0^2\frac{\partial^2}{\partial x_i^2}\right]G=\delta(t-t_0)\,\delta(\mathbf{x}-\mathbf{x}_0),\quad \frac{D}{Dt}=\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x} \quad (4.2)$$

is

$$G=\frac{\delta(t-t_0-R/c+\hat{\mathbf{r}}.\mathbf{x}_0/(1-\mathbf{M}.\hat{\mathbf{r}}))}{4\pi R(1+\mathbf{M}_0.\hat{\mathbf{r}})\,c_0^2}, \quad (4.3)$$

where $\hat{\mathbf{r}}$ is the direction of the observer relative to the source, and the result is valid in the far field.

From (4.1) the sound field is given by

$$\rho-\rho_0=\int_V G\frac{\partial^2}{\partial y_i\,\partial y_j}(H(f)\,T_{ij})\,dV-\int_S\frac{\partial G}{\partial y_i}(p_{ij}+\rho u_i(u_j-v_j))\,l_j\,dS$$
$$+\int\frac{DG}{Dt}(\rho_0 v_i+\rho(v_i-u_i))\,l_i\,dS, \quad (4.4)$$

where l_i is the normal to the surface S. In the problem under consideration, the derivatives may be taken outside the integrals to give the far-field result

$$\rho-\rho_0=\frac{1}{4\pi Rc_0^4(1+M_0\cos\theta)^3}\frac{\partial^2}{\partial t^2}\int_V[H(f)\,T_{rr}]\,dV$$
$$+\frac{1}{4\pi Rc_0^3(1+M_0\cos\theta)^2}\frac{\partial}{\partial t}\int_S[p_{rn}+\rho u_r(u_n-v_n)]\,dS$$
$$+\frac{1}{4\pi Rc_0^3(1+M_0\cos\theta)^2}\frac{\partial}{\partial t}\int_S[\rho_0 u_n+\rho(u_n-v_n)]\,dS, \quad (4.5)$$

where the square brackets signify that the integrals are to be evaluated at the retarded time $t-R/c+y_r/(1+M_0\cos\theta)$, $M_0 = U_0/c_0$, and r and n denote the radiation direction and the normal to the surface S.

We now examine the quadrupole term in more detail. The stress tensor can be written in the form $T_{rr} = (\overline{T}_{rr} + T'_{rr})\, H(g)$, where $g = 0$ is the boundary of the jet, and $\overline{T}_{rr}$ and T'_{rr} are respectively the mean and fluctuating components of T_{rr}. We now take the time derivatives inside the integral, and split g into $\bar{g} + g'$, its mean and fluctuating positions. Then

$$\frac{\partial^2}{\partial t^2} H(f)\, H(g)\, T_{rr} = \frac{\partial}{\partial t}\left(H(f) \frac{\partial T_{rr}}{\partial t} H(g) + T_{rr}\, \delta(g) \frac{\partial g}{\partial t}\right), \tag{4.6}$$

and, if g moves at velocity $\mathbf{v}$,
$$\frac{\partial g}{\partial t} + v_i \frac{\partial g}{\partial x_i} = 0,$$

so that
$$\frac{\partial^2}{\partial t^2}[H(f)\, H(g)\, T_{rr}] = \frac{\partial}{\partial t}\left(H(f) \frac{\partial T'_{rr}}{\partial t} H(\bar{g}) - \overline{T}_{rr}\, \delta(\bar{g})\, v_i \frac{\partial g'}{\partial x_i}\right), \tag{4.7}$$

where we have ignored terms of second order in the fluctuating quantities. Thus the sound due to the quadrupole sources can be written

$$(\rho - \rho_0)_{\mathrm{Q}} = \frac{1}{4\pi R c_0^4 (1 + M_0 \cos\theta)^3} \frac{\partial}{\partial t} \int \left[H(f) \frac{\partial T'_{rr}}{\partial t} H(\bar{g}) - \overline{T}_{rr} H(f)\, \delta(\bar{g})\, v_i \frac{\partial g'}{\partial x_i}\right] dV. \tag{4.8}$$

It is clear that the source term due to the steady part of T_{ij}, acting over a variable volume, is equivalent to a surface source. We may also write the sound from this quadrupole as

$$(\rho - \rho_0)_{\mathrm{Q}} = \frac{1}{4\pi R c_0^4 (1 + M_0 \cos\theta)^3} \frac{\partial}{\partial t} \int_{S_{\mathrm{J}}} [v_n \overline{T}_{rr}]\, dS_{\mathrm{J}}, \tag{4.9}$$

where S_{J} is the exterior surface of the jet, which moves at velocity $\mathbf{v}$. This is the velocity measured in free space. It is convenient to convert this into a velocity in the jet flow. To do this we write $v_n = \partial \eta / \partial t$, where η is the radial displacement of the jet boundary. Then the velocity u_n inside the jet is related to the displacement by

$$\frac{\partial \eta}{\partial t} + U_{\mathrm{J}} \frac{\partial \eta}{\partial x} = u_n,$$

so that

$$(\rho - \rho_0)_{\mathrm{Q}} = \frac{1}{4\pi R c_0^4 (1 + M_0 \cos\theta)\, (1 - M_{\mathrm{R}} \cos\theta)^2} \frac{\partial}{\partial t} \int [u_n \overline{T}_{rr}]\, dS_{\mathrm{J}}. \tag{4.10}$$

In this expression the change to the fluid velocity in the jet has caused one of the Doppler factors based on the external-flow Mach number M_0 to be replaced by one based on the relative flow velocity $(U_{\mathrm{J}} - U_0) = M_{\mathrm{R}} c_0$.

For many purposes it is useful to relate the integrand of (5.10) to the pressure and velocity in the jet, since the radial velocity of the fluid in the jet is not a quantity easily calculated in our problems. The equation of continuity is, after linearization,

$$\frac{Dp'}{Dt} + \rho_{\mathrm{J}} c_{\mathrm{J}}^2 \nabla . \mathbf{u}' = 0, \tag{4.11}$$

so that, if we consider a section of the jet flow, we find that

$$\int_V \frac{Dp}{Dt}\, dV + \rho_{\mathrm{J}} c_{\mathrm{J}}^2 \int_{S_x} u_x\, dS + \int_{S_n} \rho_{\mathrm{J}} c_{\mathrm{J}}^2 v_n\, dS = 0, \tag{4.12}$$

where S_x, S_n are the axial cross-section and the outer surface of the jet. Hence we find that for a section of the jet of length dx

$$\int_{S_x} \frac{Dp}{Dt} dS_x dx + \rho_j c_j^2 \int_{S_x} \left(\frac{\partial u}{\partial x}\right) dS_x dx + \rho_j c_j^2 \int_{S_n} v_n r d\theta dx = 0. \tag{4.13}$$

For an axisymmetric motion of the jet it follows that the 'steady' quadrupole term is

$$(\rho - \rho_0)_Q = \frac{-\overline{T}_{rr}}{4\pi R c_0^4 (1 + M_0 \cos\theta)^2 (1 - M_R \cos\theta)} \frac{\partial}{\partial t} \int \left[\frac{Dp}{Dt}\frac{1}{\rho_j c_j^2} + \frac{\partial u}{\partial x}\right] dV, \tag{4.14}$$

where the region of integration is the volume of the jet. We now apply the above results to a number of practical cases.

4.1.1. *Internal noise propagating down a pipe with internal and external flow.* We consider initially the situation described in § 3.1, and first estimate the relevant source terms. Clearly, the pressure and normal velocity on the surface of the pipe are small and zero respectively, so that the sources on the outer wall of the pipe are negligible. At the nozzle exit the pressure fluctuations are similarly negligible, as the flow cannot respond to low-amplitude fluctuations in velocity. Setting the pressure fluctuation at the nozzle equal to zero, and assuming that the radiation at low frequencies is relatively small ($O(k^2a^2)$ in energy), we find that the reflected amplitude within the pipe is -1 times the incident amplitude, in agreement with the exact solution. The axial velocity fluctuation at the nozzle is then given by $u_N = 2p'_i/\rho_j c_j$, where u_N is assumed constant across the nozzle exit.

The motion in the jet is assumed to consist of the simple convected neutrally stable wave of axial velocity fluctuation and zero-pressure fluctuation; this is the limit of the cylindrical vortex-sheet eigenfunction for very low frequencies. We now use these zeroth-order fields to evaluate the individual source terms and sound fields.

The monopole is

$$(\rho - \rho_0)_M = \frac{1}{4\pi R c_0^2 (1 + M_0 \cos\theta)^2} \frac{\partial}{\partial t} \int_S [\rho_0 v_n + \rho(u_n - v_n)] dS. \tag{4.15}$$

Here v_n is the velocity of the end of the pipe relative to the external fluid and is therefore equal to $-U_0$; the velocity u_n is then $u_N + (U_j - U_0)$, and with $\rho - \rho_j$ zero at the nozzle the monopole term is

$$(\rho - \rho_0)_M = \frac{\rho_j A_j}{4\pi R c_0^2 (1 + M_0 \cos\theta)^2} \frac{\partial u_N}{\partial t}, \tag{4.16}$$

where A_j is the exit area of the nozzle.

The dipole term is

$$(\rho - \rho_0)_D = \frac{1}{4\pi R c_0^3 (1 + M_0 \cos\theta)^2} \frac{\partial}{\partial t} \int_S [p_{nr} + \rho u_r (u_n - v_n)] dS. \tag{4.17}$$

The quantities on the nozzle exit are the same as those used for the monopole source so that $p_{nr} = 0, \rho u_r(u_n - v_n) = \rho_j (u_N + U_j - U_0)(u_N - U_j)$. Accordingly the dipole term is

$$(\rho - \rho_0)_D = \frac{\rho_j A_j (2M_R + M_0)\cos\theta}{4\pi R c_0^2 (1 + M_0 \cos\theta)^2} \left[\frac{\partial u}{\partial t}\right], \tag{4.18}$$

while the unsteady quadrupole term is

$$(\rho - \rho_0)_{UQ} = \frac{A_j}{4\pi R c_0^4 (1 + M_0 \cos\theta)^3} \frac{\partial^2}{\partial t^2} \int_0^\infty [\rho u_r u_r + (p - c_0^2 \rho)]' dy. \tag{4.19}$$

Since the motion in the jet is dominated by the instability wave, p', ρ' are zero. Then with $u = u_N + (U_j - U_0)$ we find that if $u_N \equiv u_N(t - y/U_j)$ the quadrupole term takes the form

$$(\rho-\rho_0)_{UQ} = \frac{\rho_j A_j 2M_R \cos^2\theta}{4\pi R c_0^4(1+M_0\cos\theta)^3}\int_0^\infty \frac{\partial^2}{\partial t^2}\left[u_N\left(t - \frac{y(1-(M_j-M_0)\cos\theta)}{U_j(1+M_0\cos\theta)}\right)\right]dy. \quad (4.20)$$

Integrating with respect to y gives

$$\rho-\rho_0 = \frac{\rho_j A_j 2M_R M_j \cos^2\theta}{4\pi R c_0^2(1+M_0\cos\theta)^2(1-M_R\cos\theta)}\left[\frac{\partial u}{\partial t}\right]_{y=0}^{y=\infty}, \quad (4.21)$$

and the contribution from the point at infinity must vanish for a causal solution, since the disturbance will not have reached infinity in a finite time. It follows that the contribution to the sound field from this unsteady quadrupole source is

$$(\rho-\rho_0)_{UQ} = \frac{-\rho_j A_j 2M_R M_j \cos^2\theta}{4\pi R c_0^2(1+M_0\cos\theta)^2(1-M_R\cos\theta)}\left[\frac{\partial u_N}{\partial t}\right]. \quad (4.22)$$

We now evaluate the steady quadrupole term. Since there is no pressure fluctuation in the jet this is

$$(\rho-\rho_0)_{SQ} = \frac{\overline{T}_{rr}}{4\pi R c_0^2(1+M_0\cos\theta)^2(1-M_R\cos\theta)}\frac{\partial}{\partial t}\int\left[\frac{\partial u}{\partial y}\right]dy\,dS. \quad (4.23)$$

Now the value of $\overline{T}_{rr}$ is $\rho_j\cos^2\theta(U_j-U_0)^2 - c_0^2(\rho_j-\rho_0)$, so that

$$(\rho-\rho_0)_{SQ} = \frac{\rho_j\cos^2\theta M_R^2 - (\rho_j-\rho_0)}{4\pi R c_0^2(1+M_0\cos\theta)^2(1-M_R\cos\theta)}\int\left[\frac{\partial u}{\partial x}\left(t - \frac{y(1-M_R\cos\theta)}{U_j(1+M_0\cos\theta)}\right)\right]dy\,dS; \quad (4.24)$$

and evaluating the integral as for the unsteady quadrupole, we have this sound field in the form

$$(\rho-\rho_0)_{SQ} = \frac{\rho_j A_j(M_R^2\cos^2\theta - (\rho_j/\rho_0 - 1))}{4\pi R c_0^2(1+M_0\cos\theta)(1-M_R\cos\theta)^2}\left[\frac{\partial u_N}{\partial t}\right]. \quad (4.25)$$

Adding the four source terms, we find that the total sound field is then

$$\rho-\rho_0 = \frac{\rho_j A_j}{4\pi R c_0^2}\left[\frac{\partial u_N}{\partial t}\right]\left\{\underbrace{\frac{1}{(1+M_0\cos\theta)^2}}_{\text{monopole}} + \underbrace{\frac{(2M_R+M_0)\cos\theta}{(1+M_0\cos\theta)^2}}_{\text{dipole}}\right.$$
$$\left. - \underbrace{\frac{2M_R(M_R+M_0)\cos^2\theta}{(1+M_0\cos\theta)^2(1-M_R\cos\theta)}}_{\text{unsteady quadrupole}} + \underbrace{\frac{M_R^2\cos^2\theta - (\rho_j/\rho_0-1)}{(1+M_0\cos\theta)(1-M_R\cos\theta)^2}}_{\text{steady quadrupole}}\right\}, \quad (4.26)$$

and simplification of the curly-bracketed term gives precisely the sound field obtained earlier by the Wiener–Hopf method, namely

$$\rho-\rho_0 = \frac{\rho_0 A_j}{4\pi R c_0^2(1+M_0\cos\theta)(1-M_R\cos\theta)^2}\left[\frac{\partial u_N}{\partial t}\right]. \quad (4.27)$$

For high density ratios ρ_0/ρ_j and for high Mach numbers this total field comes mainly from the *steady quadrupole term*. In particular, this is responsible for the scaling (for a given u_N) on the *far-field density* ρ_0 rather than the *jet density* ρ_j, and for the high convective amplification observed on the field shape. Further, it shows that in problems of this kind involving coupled unstable wave motion, ***it is never permissible to neglect the instability wave when calculating the sound radiation.*** Indeed the sound from these unstable (albeit neutrally stable at low frequencies) waves apparently dominates the far-field sound for high-enough Mach numbers. In some senses this last conclusion is not really surprising, as the dominance of quadrupole sources would seem to be a universal feature of high-speed flow.

We now consider, in much less detail, the radiation from a very hot jet. From the results of § 3.1 we find that all the sound energy is transmitted out of the jet pipe. The fields on the exit plane are obviously $p' = p_i$, $u = p_i/\rho_j c_j$. These give dipole and monopole sound sources as described above, and both can be neglected here since they are proportional to the jet density ρ_j, which is by assumption very small. Of the fields in the jet, that due to the propagating guided acoustic wave is very small (proportional again to ρ_j), and can be neglected. Because the density ratio is enormous, the boundary displacement is small (the jet boundary appears as if almost rigid). Therefore the steady quadrupole source is negligible. The remaining term is the quadrupole due to the pressure wave

$$p = p_i \frac{2\epsilon^2}{\pi} \exp(-\epsilon k x), \quad \epsilon^2 = \frac{1}{(\rho_0/\rho_j)(ka)^2 \ln(ka)}. \tag{4.28}$$

For the essentially illustrative purpose of this section, we consider only the low-Mach-number case. Then the quadrupole element is dominated by the term $p - c_0^2\rho$, which in the limit $\rho_0/\rho_j \to \infty$ is simply p'. Therefore the quadrupole field becomes

$$\rho - \rho_0 = \frac{1}{4\pi R c_0^4} \frac{\partial^2}{\partial t^2} \int \frac{2}{\pi} \epsilon^2 p_i \exp\left(i\omega t + iy\frac{\omega\cos\theta}{c_0} - \epsilon k y - \frac{\omega R}{c_0}\right) dy, \tag{4.29}$$

$$\sim \frac{p_i}{4\pi R c_0^4} \frac{i\omega c_0}{\cos\theta} \frac{2\epsilon^2}{\pi} \quad \text{as} \quad \epsilon \to 0. \tag{4.30}$$

Substituting for ϵ, we obtain the result

$$\rho - \rho_0 = \frac{A_j}{4\pi R} \frac{i\omega p_i}{c_0^3} \frac{2/\pi}{[(\rho_0/\rho_j)(ka)^2 \ln(ka)]}, \tag{4.31}$$

which is precisely equal to the field calculated exactly in § 3.1. We have shown further that this sound arises from the *isotropic unsteady quadrupole* term.

In the above account we have only touched on the subsonic jet with a Kutta condition. However, since the purpose of this section was mainly to illustrate the principles involved, there seems little point in proceeding with the cases of a jet with no Kutta condition or of a supersonic jet.

4.1.2. *Scattering of an externally incident sound field by a jet pipe.* This problem has been attempted by Jacques (1975) using an acoustic analogy. He, however, considered only the monopole and dipole terms on the nozzle exit. We shall show that many more source terms should be included: dipole sources on the outside wall of the pipe, and steady and unsteady quadrupole sources due to both the instability wave and the portion of the incident sound field that propagates along the jet. For simplicity we confine the analysis to a jet of the same temperature as its surroundings, and no external flow. We consider the various source terms in turn.

The amplitudes of the various sources are derived using the following low-frequency asymptotes to the unsteady flows in the jet and pipe. On the outer wall of the duct, the pressure is equal to the ambient pressure $p = p_i e^{-iku_1 x}$ for this compact jet ($ka \ll 1$). The jet itself is surrounded by a pressure fluctuation $p_i e^{-iku_1 x}$, the incident sound field, so that there is a pressure wave of this magnitude inside the jet. The incident pressure wave also sends a wave of amplitude p_i up the pipe, so that the pressure in the pipe is $p_i e^{ikx/(1-M)}$. Clearly, these pressure waves provide an imbalance in velocity on either side of the nozzle exit plane. This is balanced by the convected instability wave (which has zero pressure fluctuation) and is accordingly described by

$$u = -\frac{p_i}{\rho_j c_j}\left[1+\frac{u_1}{1+Mu_1}\right] e^{-ikx/M}. \tag{4.32}$$

We now consider each of the source terms.

The dipole source on the outside wall of the cylinder gives rise to the density field

$$(\rho-\rho_0)_{\mathrm{DW}} = \int \frac{\mathbf{n}.\hat{\mathbf{r}}}{4\pi R c_0^3}\frac{\partial}{\partial t} p_n\left(t-\frac{R}{c_0}+\frac{\mathbf{y}.\mathbf{r}}{c_0}\right) dS, \tag{4.33}$$

where S is the surface of the jet pipe. If the incident wave is of the form

$$p = p_i \exp\left[ikx\cos\theta_0 + iky\sin\theta_0\right],$$

we can write this dipole field as

$$\begin{aligned}(\rho-\rho_0)_{\mathrm{DW}} = \frac{p_i i\omega e^{-i\omega R/c_0}}{4\pi R c_0^3}\int_0^{2\pi} d\phi \int_{-\infty}^{0} 2\pi a\, dx \sin\theta\cos(\phi-\phi_0) \\ \times \exp\left[-i(\cos\theta-\cos\theta_0)kx + i(\sin\theta_0 - \sin\theta\cos\phi)ka\right],\end{aligned} \tag{4.34}$$

and to evaluate this integral for $ka \to 0$ we simply expand the exponentials for small ka. Then

$$\begin{aligned}(\rho-\rho_0)_{\mathrm{DW}} = \frac{p_i i\omega e^{-i\omega R/c_0}}{4\pi R c_0^3}\int_0^{2\pi} d\phi \int_0^{\infty} dx\, a\sin\theta\cos(\phi-\phi_0) \\ \times \exp\left[ik(\cos\theta_0-\cos\theta)x\right]\left[1+ika(\sin\theta_0-\sin\theta\cos(\phi-\phi_0))\right],\end{aligned} \tag{4.35}$$

and the only axisymmetric term is

$$\begin{aligned}(\rho-\rho_0)_{\mathrm{DW}} &= \frac{p_i i\omega\pi a}{4\pi R c_0^3} e^{-i\omega R/c_0}\int_{-\infty}^{0} ika\sin^2\theta \exp\left[i(\cos\theta_0-\cos\theta)kx\right] dx, \\ &= -\frac{p_i i\omega A_j \sin^2\theta\, e^{-i\omega R/c_0}}{4\pi R c_0^3(\cos\theta-\cos\theta_0)},\end{aligned} \tag{4.36}$$

where we have assumed that k has a small imaginary part to ensure convergence at infinity. This is the most important of the terms neglected by Jacques, and is important even for vanishingly small Mach numbers.

The monopole on the jet-pipe exit plane has strength

$$\rho u_{\mathrm{N}} = -\frac{p'}{c_0}(1-M), \tag{4.37}$$

resulting in a sound field

$$(\rho-\rho_0)_{\mathrm{M}} = \frac{i\omega A_j p_i}{4\pi R c_0^3}(1-M)\, e^{-i\omega R/c_0}. \tag{4.38}$$

The dipole strength is

$$p + (\rho u^2)' = p'(1-M)^2, \tag{4.39}$$

and, therefore, the radiation field from the dipoles on the exit of the duct is

$$(\rho-\rho_0)_{\mathrm{DE}} = \frac{i\omega A_{\mathrm{j}} p_{\mathrm{i}}}{4\pi R c_0^3}(1-M)^2 \cos\theta\, e^{-i\omega R/c_0}. \tag{4.40}$$

We consider next the unsteady quadrupole due to the instability wave; for an instability wave amplitude u_{i} this is given by

$$(\rho-\rho_0)_{\mathrm{UQI}} = \frac{i\omega A_{\mathrm{j}} \rho_{\mathrm{j}}}{4\pi R c_0^2}\frac{M^2 \cos^2\theta u_{\mathrm{i}}}{1-M\cos\theta} e^{-i\omega R/c_0}. \tag{4.41}$$

Here we have

$$u_{\mathrm{i}} = -\frac{p_{\mathrm{i}}}{\rho_{\mathrm{j}} c_{\mathrm{j}}}\left(1-\frac{\cos\theta_0}{1-M\cos\theta_0}\right), \tag{4.42}$$

so that this quadrupole field is

$$(\rho-\rho_0)_{\mathrm{UQI}} = \frac{i\omega A_{\mathrm{j}} p_{\mathrm{i}} e^{-i\omega R/c_0}}{4\pi R c_0^3}\frac{M^2\cos^2\theta}{1-M\cos\theta}\left(1+\frac{\cos\theta_0}{1-M\cos\theta_0}\right). \tag{4.43}$$

Correspondingly, the sound radiation from the steady quadrupoles excited by the instability wave is given by the previous result (5.25), with the new amplitude of the instability wave substituted and with $\rho = \rho_{\mathrm{j}}$, giving

$$(\rho-\rho_0)_{\mathrm{SQ}} = \frac{i\omega A_{\mathrm{j}} p_{\mathrm{i}} e^{-i\omega R/c_0}}{4\pi R c_0^3}\frac{M^2\cos^2\theta}{(1-M\cos\theta)^2}\left(1+\frac{\cos\theta_0}{1+M\cos\theta_0}\right). \tag{4.44}$$

The unsteady longitudinal quadrupole due to the incident wave existing in the jet flow has strength

$$((\rho u^2)'+p'-c_0^2\rho')_{rr} = p_{\mathrm{i}}' M^2\cos^2\theta + \frac{2M\cos^2\theta\cos\theta_0}{1-M\cos\theta_0}. \tag{4.45}$$

Then using the earlier results we see that the sound radiation from this source is given by

$$(\rho-\rho_0)_{\mathrm{UQJ}} = \frac{A_{\mathrm{j}} e^{-i\omega R/c_0} i\omega p_{\mathrm{i}}}{4\pi R c_0^3}\left(M^2\cos^2\theta + \frac{2M\cos^2\theta\cos\theta_0}{1-M\cos\theta_0}\right). \tag{4.46}$$

On the other hand, the steady quadrupole due to the wave in the jet has strength

$$\left(\frac{1}{\rho_{\mathrm{j}} c_{\mathrm{j}}^2}\frac{Dp}{Dt}+\frac{\partial u}{\partial x}\right)\overline{T}_{rr},$$

and the bracketed factor becomes, on substituting for p' and u,

$$\frac{i\omega p_{\mathrm{i}}}{\rho_{\mathrm{j}} c_{\mathrm{j}}}\left(\frac{\cos^2\theta_0}{1-M\cos\theta_0}-(1-M\cos\theta_0)\right),$$

while $\overline{T}_{rr}$ is again just equal to $M^2\cos^2\theta$. Then the radiation from this steady quadrupole is given by

$$(\rho-\rho_0)_{\mathrm{SQJ}} = \frac{i\omega p_{\mathrm{i}} A_{\mathrm{j}} e^{-i\omega R/c_0}}{4\pi R c_0^3}\frac{M^2\cos^2\theta(\cos^2\theta_0-(1-M\cos\theta_0)^2)}{(1-M\cos\theta)(1-M\cos\theta_0)}$$

$$\times\int_0^\infty \exp\left[ikx(\cos\theta_0-\cos\theta)\right]dx$$

$$= \frac{A_{\mathrm{j}} i\omega p_{\mathrm{i}} e^{-i\omega R/c_0} M^2\cos^2\theta(\cos^2\theta_0-(1-M\cos\theta_0)^2)}{4\pi R c_0^3(1-M\cos\theta)(1-M\cos\theta_0)(\cos\theta-\cos\theta_0)}, \tag{4.47}$$

Addition of these quantities (4.36), (4.38), (4.40), (4.43), (4.44), (4.46), (4.48) yields the radiation field derived exactly in the low-frequency limit (§ 3). Comparison of this result with Jacques' shows that he has neglected all the quadrupole sources and also the dipoles on the duct wall. In the low-Mach-number limit our radiation field is

$$\rho-\rho_0 = \frac{A_j p_i i\omega e^{-i\omega R/c_0}}{4\pi R c_0^3}\left(\cos\theta - 1 + \frac{\sin^2\theta}{\cos\theta-\cos\theta_0}\right), \tag{4.48}$$

of which the first two terms are those used by Jacques, while the last is the duct wall dipole. Adding these up gives the low-frequency low-Mach-number scattered field

$$\rho-\rho_0 = \frac{A_j p_i i\omega(1+\cos\theta_0)(1-\cos\theta)}{4\pi R c_0^3(\cos\theta-\cos\theta_0)}. \tag{4.49}$$

In this result, unlike that of Jacques, there is a reciprocal relation between the incident and scattered fields.

4.2. *The Dowling, Ffowcs Williams and Goldstein analogy*

Dowling *et al.* (1978) consider sources of sound (quadrupoles, surface dipoles and monopoles) immersed in a jet flow, and show how the acoustic analogy introduced in § 4.1 must be modified to account for both the propagation of sound through the mean flow and for the presence of flow in the acoustic environment of the source. They do this by using a non-causal Green function, free from troublesome instabilities.

Specifically, they show that for sources in a jet flow the far-field sound level is given by

$$\rho-\rho_0 = \beta\int_V \frac{\partial^2 G^+}{\partial y_i \partial y_j}(H(f)T_{ij})\,dV - \beta\int\frac{\partial}{\partial y_i}\{\delta(f)\nabla_j f(p_{ij}-\rho\mu_i(u_j-v_j))\}G^+ l_j\,dS$$
$$-\beta\int\delta(f)\nabla f\frac{DG^+}{D\tau}(\rho_0 v_i+(u_i-v_i)\rho)\,l_i\,dS. \tag{4.50}$$

In this equation G^+ is what Dowling *et al.* call the 'reciprocal Green function', representing an incoming wave (reverse-time) solution and β is $((1-M_r)^2\rho_j/\rho_0)^{-1}$, where M_r is the Mach number in the radiation direction; but they show that βG^+ is equal to the more usual Green function for a source in the jet flow with outgoing waves. In the expressions for the source strengths all the velocities and pressures are measured relative to their mean value in the medium in which they are situated.

We consider only the case of incident plane waves in the pipe. Then at the nozzle $p'=0$, and $u_i'-v_i = u_N = 2p_i/\rho_c$ while the quadrupole sources vanish since they are of second order in fluctuating quantities.

The monopole strength is then given by

$$\rho-\rho_0 = \int -\frac{DG}{D\tau}(\rho_j u_N + \rho_0(-U_0))\,dS, \tag{4.51}$$

and the second term vanishes. The axial dipole has strength $p'+\rho_j u_k(u_k-v_k)$. This is given by $\rho U_j u_N'$, so that the dipole source leads to the field

$$(\rho-\rho_0)_D = -\int\frac{\partial G}{\partial y_k}U_k\,\rho u_N\,dS. \tag{4.52}$$

Adding the two sources we note that the $U_k\,\partial G/\partial y_k$ terms cancel, leaving

$$\rho-\rho_0=\int\frac{\partial G}{\partial\tau}\rho_\mathrm{j}u'_\mathrm{N}\,dS. \tag{4.53}$$

Now for these low frequencies, it has been shown by Dowling *et al.* that with no external flow

$$G=\frac{\rho_0/\rho_\mathrm{j}}{4\pi R(1-M\cos\theta)^2}\delta\left(t-\tau-\frac{R}{c}+\frac{\mathbf{y}.\hat{\mathbf{r}}}{c}\right). \tag{4.54}$$

Substitution of this in (4.53) leads to the far-field density fluctuation

$$\rho-\rho_0=\frac{\rho_0A_\mathrm{j}\,\partial u_\mathrm{N}/\partial t}{4\pi Rc_0^2(1-M\cos\theta)^2}. \tag{4.55}$$

This result is valid for no external flow. When external flow is present, the only change is that the Green function is multiplied by $(1+M_0\cos\theta)^{-1}$ and the original result is quickly recovered.

In applying this analogy which explicitly incorporates a mean flow we have removed the quadrupole sources, which are now included implicitly in the Green function, which then accounts for all propagation effects. We have given only this one example for the purpose of illustration. The sound fields for the other cases discussed earlier could be derived with equal facility using this analogy. In particular, the light-jet result follows easily if the appropriate Green function is used.

5. The effects of nozzle contraction

In this section we examine the change in reflection coefficient and sound radiation (§ 2) when a contracting nozzle is connected to the pipe. Additionally we determine the radiation produced when a slug of fluid of different entropy from the mean flow convects through the duct.

The method of analysis we use is to assume that the nozzle is sufficiently short that the flow through it is quasi-static with no instantaneous storage of mass or energy in the nozzle. We need therefore only consider the conservation of mass flow or energy flux across the nozzle. Our method is then identical to that employed by Cumpsty & Marble (1977) for turbine disks and by Marble & Candel (1977) for variable-area ducts. It is also similar to an analysis of the nozzle problem by Ffowcs Williams (1972). That analysis, though, contains an error (see Mani 1981). We further assume that at these low frequencies the boundary condition at the end of the nozzle is that the pressure fluctuation p' is zero (cf. § 2). For higher frequencies the theory could still be used but some sort of impedance condition at the nozzle exit would have to be used.

The equation of continuity of mass flow, applied at the two ends of the nozzle, at stations 1 and 2, say, is $(\rho UA)_1-(\rho UA)_2=0$. Linearizing this in the fluctuations in density and velocity gives

$$\frac{\rho'_1}{\rho_1}+\frac{u'_1}{U_1}=\frac{\rho'_2}{\rho_2}+\frac{u'_2}{U_2}, \tag{5.1}$$

an equation exact for low-enough frequencies. At higher frequencies it should be augmented by a term

$$\int_1^2\frac{A}{\rho UA}\frac{\partial\rho}{\partial t}dx$$

representing the instantaneous storage of mass in the nozzle. For a frequency ω, this term is of order $\omega L/c$ smaller than the others, where L is a typical nozzle length, and may be neglected here. (Of course the argument is only valid for fixed values of M, and in particular is not expected to be uniformly valid as $M \to 1$.) Since $p_2' = 0$ and entropy is conserved, (5.1) may be rewritten

$$\left(\frac{p'}{\rho c^2}+\frac{u'}{U}\right)_1 = \left(\frac{u'}{U}\right)_2. \tag{5.2}$$

The other equation we use is the energy equation. This states that across the nozzle the specific stagnation enthalpy is conserved, so that

$$(C_p T' + Uu')_1 = (C_p T' + Uu')_2, \tag{5.3}$$

where T' is the temperature fluctuation. Since entropy S is conserved and

$$T\,dS = C_p dT - dp/\rho,$$

it follows that, with $p_2' = 0$ and $s_2' = s_1'$,

$$\frac{p_1'}{\rho_1} + U_1 u_1' + (T_1 - T_2)\,s_1' = U_2 u_2'. \tag{5.4}$$

In this equation, as with the continuity equation, we have neglected a term of relative order $\omega L/c$ representing the unsteady storage of energy in the nozzle.

We now assume that upstream of the nozzle there are incident and reflected waves $p_1 e^{-i\omega x/(U_1+c_1)}$ and $Rp_1 e^{i\omega x/(c_1-U_1)}$. Downstream of the nozzle there is a convected neutrally stable wave $u_2 e^{-i\omega x/U_2}$. We substitute these forms into our mass-flow and energy-conservation equations giving respectively

$$\frac{p_1}{\rho_1 c_1^2}[(1+R)M_1+(1-R)] = \frac{u_2}{c_2}\frac{M_1}{M_2}, \tag{5.5}$$

$$\frac{p_1}{\rho_1 c_1^2}[(1+R)+M_1(1-R)] = M_2\frac{c_2^2}{c_1^2}\frac{U_2}{c_2}. \tag{5.6}$$

Solving (5.5) and (5.6), we find that the velocity is

$$u_2 = \frac{2M_2 c_2}{M_1 c_1}\frac{1+M_1}{1+M_2^2c_2^2/M_1c_1^2}\frac{p_1}{\rho_1 c_1}, \tag{5.7}$$

and the reflection coefficient is

$$R = -\frac{1+M_1}{1-M_1}\frac{1-M_2^2c_2^2/M_1c_1^2}{1+M_2^2c_2^2/M_1c_1^2}. \tag{5.8}$$

In these expressions, we can use

$$\frac{c_2^2}{c_1^2} = \frac{1+\frac{1}{2}(\gamma-1)M_1^2}{1+\frac{1}{2}(\gamma-1)M_2^2} \tag{5.9}$$

for isentropic flow, γ here denoting the adiabatic exponent, while for small Mach numbers, $M_2/M_1 = A_1/A_2$ (the area ratio of the nozzle), so that then

$$R = -\frac{1+M_2A_2/A_1}{1-M_2A_2/A_1}\frac{1-M_2A_1/A_2}{1+M_2A_1/A_2}. \tag{5.10}$$

It is clear from this expression that *the reflection coefficient is zero when* $M_2 c_2^2/M_1 c_1^2 = 1$, that is, when $M_2 = A_2/A_1$ for low-enough M_2. The result is in agreement with the recent experimental results of Bechert (1979). In that paper Bechert presents a theory for this phenomenon which is similar to ours, except that it does not include the effects of compressibility, and is therefore restricted to low Mach numbers.

A consequence of the above theory is that since both the radiation field and the instability-wave amplitude depend only on the velocity u_2 at the nozzle exit, the ratio of their net energy fluxes is unchanged and quite independent of the nozzle contraction. It is nevertheless of interest to express the radiated sound in terms of the upstream pressure wave p_I. The radiated sound power is

$$W_\mathrm{R} = \frac{\rho_0 \omega^2 a^2 A_\mathrm{J} u_\mathrm{N}^2 (1 + \frac{1}{3} M_\mathrm{R}^2)}{4 c_0 (1 - M_\mathrm{R}^2)^3}. \tag{5.11}$$

Substituting for $u_\mathrm{N} = u_2$, this becomes

$$W_\mathrm{R} = \frac{1 + \frac{1}{3} M_\mathrm{R}^2}{(1 - M_\mathrm{R}^2)^3} \left(\frac{\omega a}{c_0}\right)^2 \frac{\rho_0 c_0 c_2}{\rho_2 c_2 c_1} \frac{W_\mathrm{I}}{1 + M_2^2 c_2^2/M_1 c_1^2}, \tag{5.12}$$

where W_I is the power flux in the incident wave in the pipe. Therefore the ratio $W_\mathrm{R}/W_\mathrm{I}$ of the far-field to the incident power is increased in the ratio $c_2/c_1(1 + M_2^2 c_2^2/M_1 c_1^2)$ by the contraction. This ratio is less than unity, which shows that there is always less power radiated owing to the addition of nozzle contraction, even at the condition when the reflection coefficient is zero. In that case, all the incident power is, to first order, transferred to the *instability wave*.

We consider next the transmission of sound out of a choked nozzle. Instead of assuming as the boundary condition that there is an instability wave downstream with zero pressure, we use a condition of constant non-dimensional mass flow through the choked nozzle. This condition is the same as that introduced by Cumpsty & Marble for a choked turbine. The choked nozzle condition is that

$$m T_{01}^{\frac{1}{2}} / A p_{01} = \text{constant},$$

where m is the mass flow, A the area, and T_{01} and p_{01} the stagnation temperature and pressure. In this case the energy equation cannot be used to determine the unsteady flows, since the choked flow is not isentropic.

Cumpsty & Marble linearize the constant-mass-flow condition to obtain an extra equation relating the pressure, temperature and velocity at the entrance to the nozzle. In our case there is no need to do this. We note that, since both the choked and subsonic values of the reflection coefficient must be the same when $M_2 = 1$, we can obtain the choked-flow reflection coefficient for arbitrary upstream Mach number M_1 by simply setting $M_2 = 1$ in (5.8). Then with $c_2^2/c_1^2 = (1 + \frac{1}{2}(\gamma - 1) M_1^2)/\frac{1}{2}(\gamma + 1)$, the reflection coefficient is

$$R = -\frac{(1 + M_1)((\gamma + 1) M_1 - 2 - (\gamma - 1) M_1^2)}{(1 - M_1)((\gamma + 1) M_1 + 2 + (\gamma - 1) M_1^2)} \tag{5.13}$$

$$= \frac{1 - \frac{1}{2}(\gamma - 1) M_1}{1 + \frac{1}{2}(\gamma - 1) M_1}. \tag{5.14}$$

For subsonic M_1 (which is always the case) this reflection coefficient is always positive and less than unity. This may be compared with the negative value obtained for a non-contracting nozzle. If the full analysis with the constant-mass-flow relation is used, the same result is obtained.

Another interesting result that can be obtained from the above theory is the reflection coefficient of a duct inlet. This result is obtained by reversing the sign of the Mach numbers in the formula (setting $M_1 = -M_2$, $M_2 = -M_1$) and putting $A_2/A_1 = 0$, for a 'bellmouth' inlet. Then we find that the magnitude of the reflection coefficient is equal to $(1-M)/(1+M)$. This is in good agreement with the experimental value of Ingard & Singhal (1975), who obtain a value of $[(1-M)/(1+M)]^{1.33}$. Further, it corresponds to total reflection of the sound energy incident on the end of the tube.

Finally, consider an entropy wave incident on the nozzle from upstream. The detailed analysis, which proceeds in a manner similar to that for the reflection coefficient, may be found in Cargill (1981). The upshot is a radiation field that, retaining terms $O(M)$, is given by (with $\rho_2 = \rho_0$, $c_2 = c_0$)

$$p_2' = \frac{1+2M\cos\theta}{4\pi R} A_2\rho_0 \frac{\partial}{\partial t}\left[\frac{\Delta\rho\Delta PA_1}{\rho_1^2 c_1 A_2(1+MA_1/A_2)}\right], \tag{5.15}$$

which is in precise agreement with the results derived by more sophisticated means by Ffowcs Williams & Howe (1975). Their analysis assumes that a sharp-fronted slug of gas of density different from that of the mean flow is convected through the nozzle, and determines the far-field sound by a rather more elegant application of the acoustic analogy.

6. Discussion and comparison with experimental results

The purpose of this section is to discuss the overall features of the results obtained in §§ 2–5 and to compare them with such experimental results as are available.

There are a number of comparisons with published data that can be made for incident internal noise. Figure 5 compares our low-frequency field shapes (2.45) with the exact calculations of Munt (1977) for the same problem, for cold jet conditions ($C = 1$); the two agree beyond 60° to the jet axis. Near the jet axis there is a discrepancy which increases with frequency and Mach number. This might have been expected since our predicted power levels increase very rapidly as M nears unity, and would be expected to exceed the exact values. We note that, in the theory, as the jet nears sonic velocity one of the branch points tends to infinity and then the approximate factorizations which we have used are not uniformly valid as $M \to 1$. That would accord with expectations that the reflection coefficient should actually decrease near $M = 1$, so that at $M = 1$ it changes gradually to its zero value for a supersonic jet. Therefore our solution is expected to be invalid for Mach numbers close to one. In Munt's (1977) paper theory is compared with the experimental results of Pinker & Bryce (1976) for both hot and cold jets. In the latter case the agreement is good, as it is for our theory for low-enough Mach numbers. For the hot jet Munt's results are much lower than the experimental points close to the jet axis, and show a dip consistent with refraction of sound by the jet. A possible reason for this disagreement is the incomplete modelling of the jet instability waves. In the model problem these grow exponentially as along the jet and have no conventional acoustic far field. In reality, however, the growth is limited by the spreading of the mean flow downstream of the

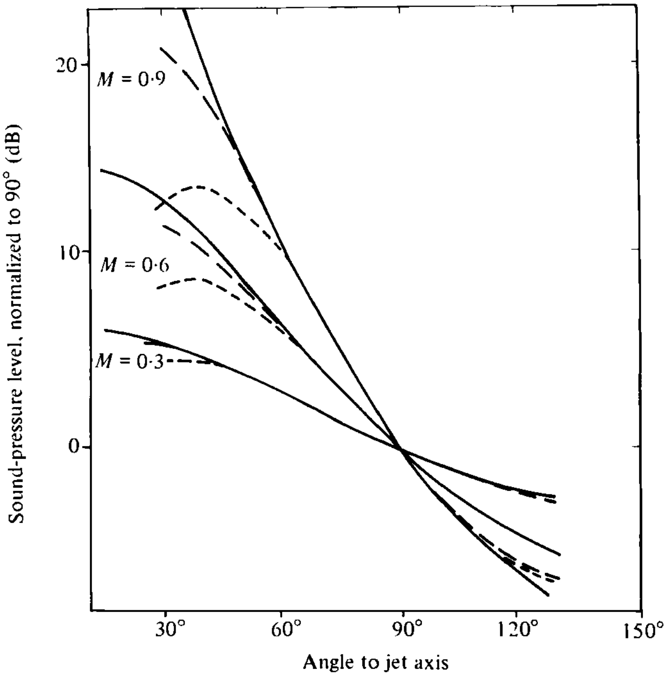

FIGURE 5. Comparison of low-frequency field shapes with Munt's calculations; $\alpha = 0$, $C = 1{\cdot}0$. ——, $ka \to 0$; — —, $ka = 0{\cdot}24$ (Munt 1977); – – –, $ka = 0{\cdot}6$ (Munt 1977).

nozzle and by nonlinear effects. Further, in Munt's theory the region of the discrepancy is the one where the direct field of the instability is present, and limiting the growth of this instability would probably result in an extra far field, dependent on the growth and decay rates of the instability wave, but confined essentially to the angular region in which the direct field of the original instability wave was present. In our theory, there is no such far field outside the jet associated with the instability waves, since this angular sector is vanishingly small for these low-frequency waves which grow at negligible rate.

The reflection coefficient we have determined is in agreement with both the limited experimental data of Schlinker (1977) and Munt's (1982*a*) computations. However, it would appear to be valid over only a limited frequency range. At non-zero frequency it is found that for non-zero Mach numbers the reflection coefficient initially rises, to give a peak at a nearly constant Strouhal number, and then decreases as more sound is radiated, in accordance with the established theory without flow (Levine & Schwinger 1948). We note though that this behaviour does not violate conservation of energy, since $|R|$ is always less than $(1+M)/(1-M)$. In another paper (Cargill 1982) we carry out the low-frequency calculation initiated here to higher order in ka, with results that adequately reproduce the entire behaviour observed experimentally and computed by Munt.

In our theory the effect of external flow on intensity has been shown to vary nearly as $(1+M_0\cos\theta)^{-6}$ near $\theta = 90°$. This is in excellent agreement with the results of Pinker & Bryce (1976), which covered higher frequencies. The highly directional field-shape we obtain is, further, characteristic of sources immersed in jet flows at low frequency (Goldstein 1975; Mani 1974).

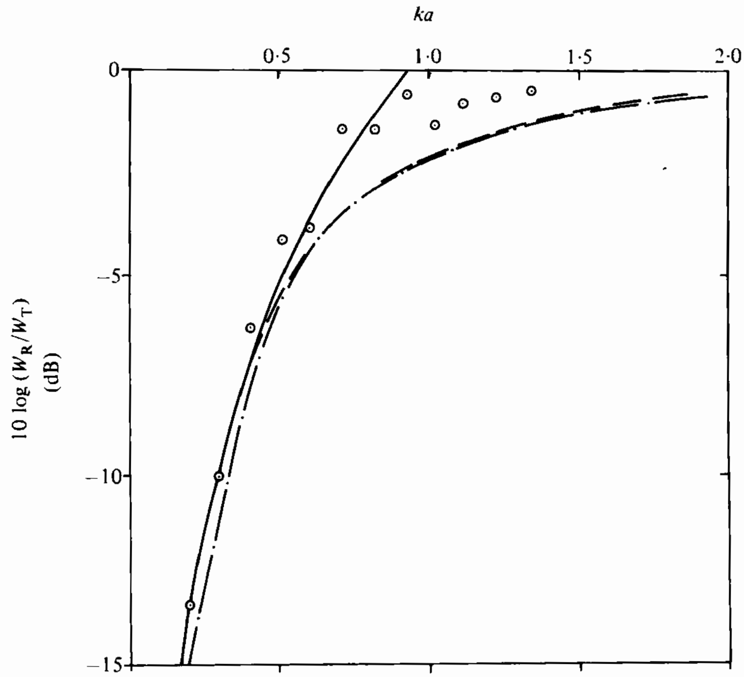

FIGURE 6. Comparison of the ratio of radiated power to net power in a duct with measurements and Munt's theory; $M = 0{\cdot}3$, $\alpha = 0$, $C = 1{\cdot}0$. — —, exact theory (Munt 1982*b*), ——, low-frequency theory; — · —, approximate theory (Howe 1979); ⊙, measurements (Bechert *et al.* 1977).

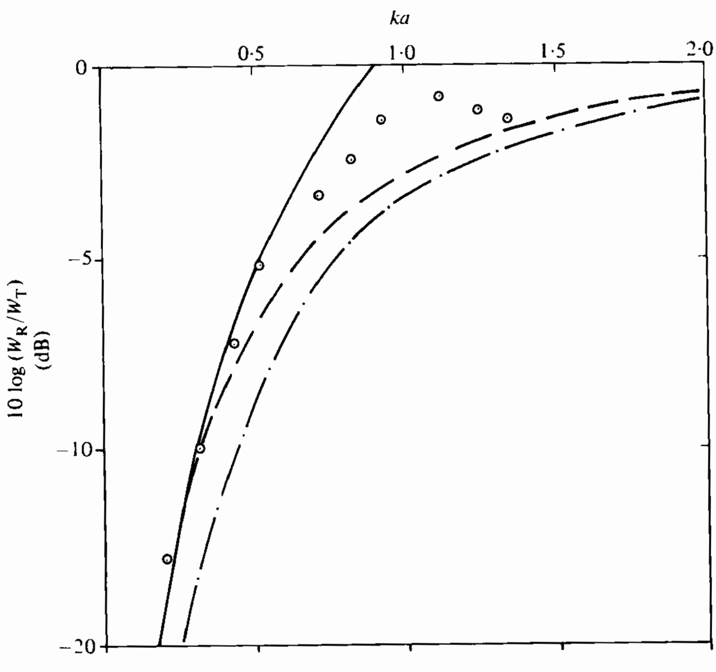

FIGURE 7. Comparison of the ratio of radiated power to net power in a duct with measurements and Munt's theory; $M = 0{\cdot}5$, $\alpha = 0$, $C = 1{\cdot}0$. — —, exact theory (Munt 1982*b*); ——, low-frequency theory; — · —, approximate theory (Howe 1979); ⊙, measurements (Bechert *et al.* 1977).

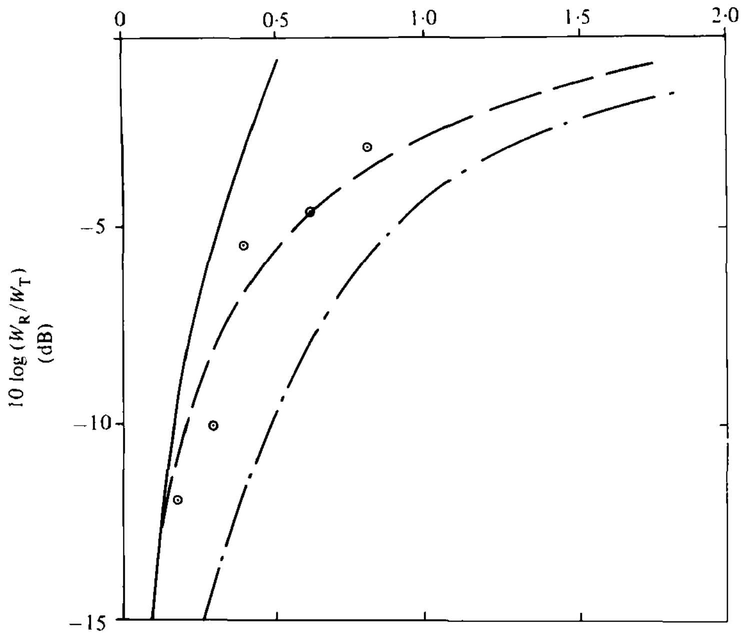

FIGURE 8. Comparison of the ratio of radiated power to net power in a duct with measurements and Munt's theory; $M = 0{\cdot}7$, $\alpha = 0$, $C = 1{\cdot}0$. — —, exact theory (Munt 1982*b*); ——, low-frequency theory; — · —, approximate theory (Howe 1979); ○, measurements (Bechert *et al.* 1977).

Of great interest is the comparison between the net power in the pipe and the power radiated to the far field. Figures 6–8 compare our results with Munt's exact theory (1982*b*). Howe's low-Mach-number theory (1979) and the experiments of Bechert *et al.* (1977). For the lowest frequencies all four are in good agreement. As might be expected, our theory diverges from the experiments and Munt's theory for higher frequencies, and agreement is only obtained over reduced frequency ranges as the Mach number is increased, which is consistent with overprediction of the far-field sound levels. We have further shown that the conversion from acoustic to hydrodynamic energy implicit in these relations is critically dependent on the existence of a Kutta condition at the pipe exit. When the Kutta condition is relaxed, and no jet instability wave is produced, we find that there is no such energy conversion, in agreement with Howe (1979). Further, we find that then all the incident energy is reflected up the duct and the reflection coefficient is $-(1+M)/(1-M)$. We have also shown, again in agreement with Howe, that, if the instability wave is replaced by some sort of neutral wave convected at a speed $\nu M c_j$, then the radiation changes with ν from the Kutta-condition value ($\nu = 1$) to the non-Kutta-condition value ($\nu = 0$).

An alternative way of looking at the power transmission ratio is as a function of Mach number. In figure 9 we compare our results with those of Moore (1977). We find that at Mach numbers between 0·2 and 0·8 agreement is good, despite the relatively high *ka* value (0·46) of Moore's experiments. At low Mach numbers our result fails because the Strouhal number of his experiment is no longer low, while at high Mach numbers we probably overestimate the far-field radiation.

A further corollary to this energy-loss mechanism concerns the resonances in a tube with flow. We have shown that energy is lost from such a tube, and this loss would

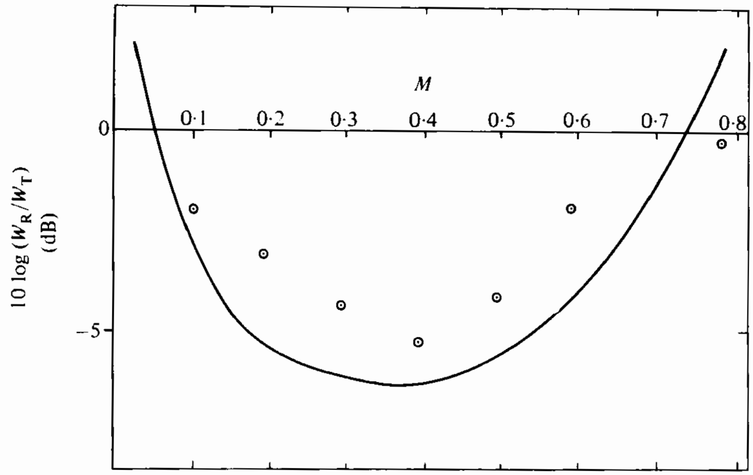

FIGURE 9. Comparison of the ratio of radiated power to net power in a duct with measurements by Moore (1977); $\alpha = 0$, $C = 1{\cdot}0$, $ka = 0{\cdot}46$. ——, low-frequency theory; ⊙, measurements.

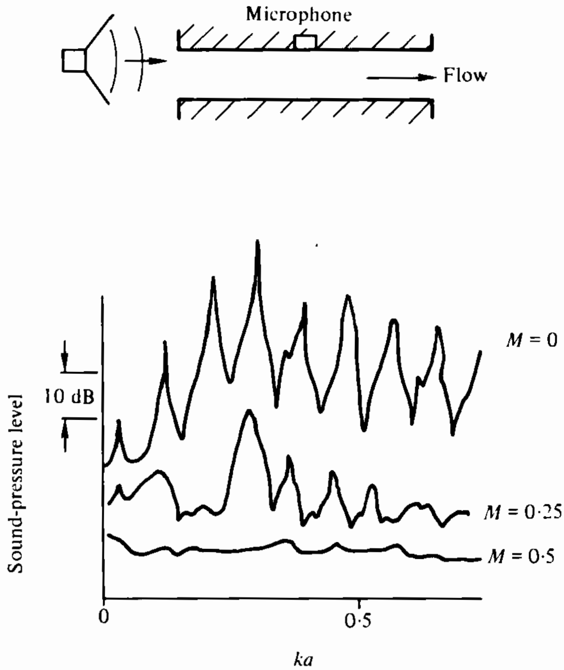

FIGURE 10. Effect of flow on duct resonances (after Ingard & Singhal 1975).

result in the elimination of any resonant peaks. This has been demonstrated by Ingard & Singhal (1975). Their results, as reproduced in figure 10, do indeed show a significant reduction in the relative amplitude of the resonant peaks of the frequency response when a mean flow is present.

When the jet is 'hotter than it is compact' we find that a quite different set of phenomena occurs. Then, all the sound escapes from the pipe (the reflection coefficient is zero) and is channelled along the jet, which in this limit behaves as a rigid walled tube. There is no jet instability wave. Further, the pressure in the far field is reduced relative to its normal ($\rho_0/\rho_j \sim O(1)$) value by a factor $\sim [(\rho_0/\rho_j)(ka)^2 \ln(ka)]^{-1}$. This factor is by definition large in the light-jet condition. We find, though, that for a jet composed of a perfect gas, the condition always fails around the 90° position in the far field, where there is a peak in the field shape corresponding to the Mach angle for disturbances transmitted along this very hot jet. These results are entirely consistent with those established by Dowling *et al.* (1978) for jet noise. Interesting though this result is, it appears to have little relevance in an aeronautical context, as the temperatures required to achieve the light-jet condition are far too high ($\sim$ 10000 K).

Examination of our results for a supersonic jet shows phenomena similar to those for the subsonic jet. Again there is a conversion from acoustic to hydrodynamic energy. But, compared with the subsonic jet, the reflection coefficient is now zero, since sound cannot propagate upstream against the flow, and there is an additional motion of the jet which corresponds to the steady wave structure of an imperfectly expanded supersonic jet. The energy in the pipe splits itself between the instability wave and these quasi-periodic waves. The field shape of the radiated sound is also somewhat changed as compared with the subsonic case.

Our result for the scattering of an externally incident sound wave by the pipe may be compared with the theory of Jacques. He deduces the radiated sound from an application of the acoustic analogy. We show this to be incorrect, firstly because he neglects the sources on the wall of the pipe, and secondly because he neglects the quadrupole sources in the jet. Our results do, however, agree with his for the 'zeroth-order' fields in the pipe and jet column. An interesting feature of the field shape of the radiated sound is the appearance of a zero at the cone-of-silence angle for waves propagating out of the jet and into the ambient fluid.

A further problem that can be handled using the methods of this paper is the generation of sound when turbulence is convected past the end of the jet pipe. This is described in detail in Cargill (1981), and we will only summarize the results here. The turbulence is modelled by convected ring vortices which may be situated inside or outside the pipe. This sound is shown to scale as $p^2 \propto \rho^2 U^4 M^2 l^2/R^2$, which is in agreement with other theories, for example that of Leppington (1971), who modelled turbulence by point quadrupoles. The sound source due to convection of vortices past the end of the pipe only exists when there is an external flow over the jet, and could be one of the 'installation effects' of Bryce (1979) which raise the noise level of an aircraft in flight above the level predicted for pure jet noise. An important feature of our result is that, when a Kutta condition is enforced, no sound is radiated when the vortices are convected at the speed of the mean flow. This is similar to a result obtained by Howe (1976) for the convection of line vortices past a flat plate. In our model it arises because the sound field is essentially driven by the pressure that would exist on the wall of the duct if it were infinite, and in our linear approximation this is proportional to the convection speed of the vortices relative to the mean flow. When no Kutta conditions are enforced, the response of the sound field to this pressure is increased and the dependence on the velocity of slip removed.

In Cargill (1981) we have re-examined Crighton's (1972) theory for the scattering of an instability wave by the pipe. We find a result that agrees with his in the zero-Mach-number limit, but differs somewhat otherwise, where the field shape is altered owing to the internal and external flows. Then the effect of flight is more complicated than the four powers of Doppler factor assumed by Crighton.

In all these problems which we have solved by the Wiener–Hopf method in the low-frequency limit, we have implicitly assumed that both Strouhal number ka/M and Helmholtz number are small. This limits the usefulness of the solutions. In an aeronautical context, Strouhal numbers of order one are important. Examination of all our formulae shows, however, that as the frequency is changed the field shapes are all changed by the same factor ($\sim 1/K^{+}(u)$). To obtain the behaviour at these higher frequencies all we have to do, therefore, is use Munt's results for the internal noise at higher frequency, and scale the other results appropriately. Subject to the comments we have already made about Munt's results compared with ours, our results for these other mechanisms may be directly read across to higher frequencies.

We have used Lighthill's acoustic analogy to deduce a set of equivalent sources for these sound fields, and we find that there are usually four types of source: dipoles and monopoles on the duct exit and side walls, and two types of quadrupole in the jet flow. The quadrupoles involve the unsteady part of the Lighthill stress tensor acting over a fixed volume, and the steady part of the stress tensor acting over the variable volume of the jet, the latter reducing to a surface source on the outer surface of the jet. At higher Mach numbers and for high density ratios, the sound from the steady quadrupole dominates the far field and is responsible for the high convective amplification on the field shape of internal noise radiation. It is also responsible for the sound field being proportional not to the jet density as one might expect, but to the far-field density (for a given velocity fluctuation at the nozzle exit). We have also shown, in consequence, that in problems such as these *the instability wave is an essential feature of the unsteady motion of the jet.* In the low-frequency limit, the instability wave degenerates to a *neutral* convected vorticity pattern on the jet boundary.

We have also used another analogy, due to Dowling *et al.* (1978), which incorporates explicitly the effects of fluid shielding by the mean flow. Then the only sources are those dipoles and monopoles on the duct exit alone, while the quadrupole sources are negligible, being now of second order in fluctuating quantities. Thus the field shape and density dependence appear as an artifact of the particular Green function used and not of the quadrupole sources.

We have produced a simple theory for the effects on these sound radiation problems of the contraction of the nozzle. In the low-frequency limit we find that this contraction has no effect on the transfer of power from acoustic to hydrodynamic energy, but does have a large effect on the reflection coefficient. Indeed, as the Mach number increases from zero, the reflection coefficient decreases instead of remaining constant, reaching zero when the Mach number is equal to the area ratio of the nozzle. This behaviour is found in recent experimental results of Bechert (1979), and figure 11 compares our result with his. The position of the minimum in the reflection coefficient is well predicted. For a choked supersonic nozzle we find the reflection coefficient is always positive, and less than unity.

We have also used this theory to study the sound produced where 'hot spots' or entropy waves are convected out of the nozzle. Our results are in excellent agreement, for low Mach number, with those of Ffowcs Williams & Howe (1975), and show the sound field to depend on the temperature drop across the nozzle.

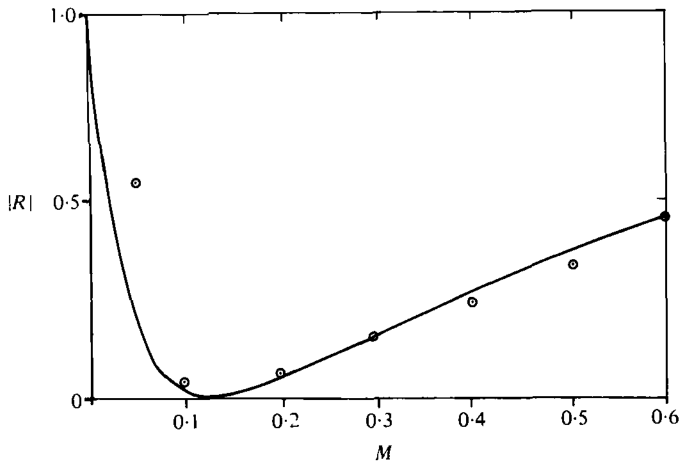

FIGURE 11. Sound-power reflection coefficient: comparison with Bechert (1979). ——, low-frequency theory; ⊙, Bechert.

The author wishes to thank his employers (Rolls-Royce Limited) for permission to publish this paper, and the Science Research Council for an Industrial Studentship. He also wishes to acknowledge the extensive and helpful advice given by his supervisor Professor D. G. Crighton.

Appendix A. Properties of the Wiener–Hopf kernel: subsonic flow

The purpose of this appendix is to set out the properties of the kernel $K(u)$ of the Wiener–Hopf equation (2.20):

$$K(u) = \rho_j c_j k^2 \frac{[D_j^2 J_m(kva)\, kwH_m^{(2)\prime}(kwa) - \gamma D_0^2 H_m^{(2)} kwa)\, kvJ'_m(kva)]}{kvkwJ'_m(kva)\, H_m^{(2)\prime}(kwa)}. \quad \text{(A 1)}$$

We consider first the axisymmetric case, $m = 0$. Then as $ka \to 0$ the denominator

$$kvkwJ'_m(kva)\, H_m^{(2)}(kwa) \sim \frac{i}{\pi} k^2 v^2. \quad \text{(A 2)}$$

This has the factorization $(ik^2/\pi)\, v^+ v^-$, where

$$v^+ = (1-Mu)-u, \quad v^- = (1-Mu)+u. \quad \text{(A 3}a, b)$$

The quantity

$$D_j^2 J_m(kva)\, kwH_m^{(2)\prime}(kwa) - \gamma D_0^2 H_m^{(2)}(kwa)\, vkJ'_m(kva) = Q \quad \text{(say)}$$

is, to second order in ka,

$$Q = \frac{2iD_j^2}{\pi a}\left[1 - \frac{1}{2}\left[\frac{D_j^2(kwa)^2 - \gamma D_0^2(kva)^2}{D_j^2}\right][\ln \tfrac{1}{2}kwa + \gamma_E - \tfrac{1}{2}\pi i - \tfrac{1}{2}]\right], \quad \text{(A 4)}$$

where Euler's constant $\gamma_E = 0{\cdot}577\,21\ldots$. The zeros of this expression depend on the ranges of the parameters involved. We distinguish between the two cases, that in which γ is $O(1)$ as $ka \to 0$, and the light-jet case of Dowling *et al.* (1978), where

$\gamma \gg 1/(ka)^2 \ln ka$ as $ka \to 0$. For the former case the zeros are near $u = 1/M$, at $u = u_0, u_0^* = (1/M)(1 \pm i\sigma)$, where

$$\sigma = (\tfrac{1}{2}\gamma)^{\frac{1}{2}} D_0 kva[\ln \tfrac{1}{2}wka + \gamma_E - \tfrac{1}{2} - \tfrac{1}{2}\pi i], \tag{A 5}$$

which is to be evaluated with $u = 1/M$. It is then clear that Q may be factorized as Q^+Q^-, where, for $k \in \Delta$ (figure 2),

$$Q^+ = \frac{2iM^2}{a\pi}(u-u_0)(u-u_0^*), \quad Q^- = 1, \tag{A 6}$$

with

$$u_0, u_0^* = \frac{1}{M} \pm \frac{i\gamma^{\frac{1}{2}}(1-\alpha)}{\sqrt{2}\,M^2}\left[\ln\left(\frac{ka}{2M}(1-C^2(1-\alpha)^2M^2)^{\frac{1}{2}}\right) + \gamma_E - \tfrac{1}{2}\pi i - \tfrac{1}{2}\right]. \tag{A 7}$$

This expression differs from Munt's, because we have included terms of $O(k^2a^2)$ and not just $O((ka)^2 \ln ka)$ to obtain the correct normalization for the $\ln ka$ term.

When γ is sufficiently large it is clear that σ is no longer small, and this approximation breaks down. This is the light-jet limit. There, the second term in Q dominates. In Q, we have $\gamma = C^2$ for a perfect gas, and therefore

$$D_j^2(kwa)^2 - \gamma D_0^2(kva)^2 = -(kau)^2(D_j^2 - \gamma D_0^2), \tag{A 8}$$

$$Q = \frac{2i}{\pi a}(D_j^2 + \tfrac{1}{2}k^2a^2u^2(D_j^2 - \gamma D_0^2)(\ln \tfrac{1}{2}kua + \gamma_E - \tfrac{1}{2}i\pi - \tfrac{1}{2}) + O(k^3a^3). \tag{A 9}$$

Then the zeros of Q are near $u = 0$, at $\pm i\epsilon$, say, where ϵ satisfies $Q(\pm i\epsilon) = 0$, or

$$1 + k^2a^2\epsilon^2\gamma[\ln \tfrac{1}{2}kaC + \gamma_E - \tfrac{1}{2}\pi i - \tfrac{1}{2}] = 0, \tag{A 10}$$

so that to a first approximation

$$\epsilon = \sqrt{2}/\gamma^{\frac{1}{2}}ka|\ln \tfrac{1}{2}kaC|^{\frac{1}{2}}. \tag{A 11}$$

Then the factorization is

$$Q^+ = \frac{2i}{\pi a\epsilon^2}(u+i\epsilon), \quad Q^- = (u-i\epsilon). \tag{A 12}$$

We now consider the case of other azimuthal modes, at low frequency. In the limit of small ka, u finite, the mth azimuthal kernel function $K_m(u)$ may be expanded for small ka to give

$$K_m(u) \sim \frac{\rho_j c_j^2 k^2a^2\left[D_j^2\left(\frac{kva}{m!}\right)(\frac{1}{2}kva)^m \frac{i}{2\pi}\left(\frac{2}{kwa}\right)^{m+1} + \gamma D_0^2 \frac{m}{m!}(\frac{1}{2}kva)^{m-1}\frac{1}{2}kva\frac{i}{\pi}\left(\frac{2}{kwa}\right)^m\right]}{kva\,kwa\,\frac{im}{2\pi}\left(\frac{2}{kwa}\right)^{m+1}\frac{1}{2}\frac{m}{m!}(\frac{1}{2}kva)^{m-1}}$$

$$= \rho_j c_j^2 k^2a^2(D_j^2 + \gamma D_0^2). \tag{A 13}$$

The zeros of this factor are both in R^-, so that we obtain the result

$$\left.\begin{aligned} K_m^+ &= \rho_j c_j^2(ka)^2(D_j^2 + \gamma D_0^2), \\ K_m^- &= 1. \end{aligned}\right\} \tag{A 14}$$

This factor $D_j^2 + \gamma D_0^2$ will be recognized as the dispersion relation describing the instabilities of a plane two-dimensional vortex sheet in compressible flow. The zeros are where

$$(1-Mu)^2 = \pm i\gamma(1-M\alpha u)^2, \tag{A 15}$$

i.e. at
$$u_0, u_0^* = \frac{1}{M}\frac{1 \pm i\gamma}{1 \pm i\alpha\gamma}, \tag{A 16}$$

where the upper and lower signs refer to the stable and unstable modes of the jet. The factor $D_j^2 + \gamma D_0^2$ may then be written as

$$D_j^2 + \gamma D_0^2 = M^2(1+\alpha^2\gamma^2)(u-u_0)(u-u_0^*). \tag{A 17}$$

Appendix B. Properties of the Wiener–Hopf kernel: supersonic flow

This appendix examines the properties of the Wiener–Hopf kernel $K(u)$ for supersonic conditions. As before, $K(u)$ is given by

$$K(u) = \frac{\rho_j c_j^2 k^2 [D_j^2 J_m(kva)\, kwH_m^{(2)\prime}(kwa) - \gamma D_0^2 H_m^{(2)}(kwa) kvJ_m'(kva)]}{kvkwJ_m'(kva)\, H_m^{(2)\prime}(kwa)}. \tag{B 1}$$

For convenience we consider only the $m = 0$ mode and ignore the light-jet condition.

In the subsonic case, the only poles of the numerator of (B 1) that were important at low frequencies were those representing instability waves. The other poles, near the zeros of $J_m(kva)$, represented waves in the jet decaying as $\exp[-j_{mn}/a(1-M^2)^{\frac{1}{2}}]$, where $J_m(j_{mn}) = 0$, and were unimportant. For supersonic jet speeds, these poles produce non-decaying waves, which are the analogue of the wave structure of an imperfectly expanded jet in steady flow. We divide the range of u into two regimes for the factorization of $K(u)$. First, where $u \ll 1/ka$ these poles are of no consequence, and we can again approximate $K(u)$ as

$$K(u) = \frac{-2\rho_j c_j^2 k^2 D_j^2}{(kv)^2 a}. \tag{B 2}$$

For supersonic flow this is a plus function. This is because $K(u)$ depends only on the jet, not ambient, conditions, and because no waves can propagate against the flow. Therefore we can take $K^-(u) = 1$ and $K^+(u) = K(u)$.

For values of $u \gg 1/ka$ the poles of the numerator of (B 1) become significant. We only deal with the case of no external flow, where $\alpha = 0$. Then, we can approximate $K(u)$ as

$$K(u) = \frac{2\rho_j c_j^2 k^2 D_j^2 J_m(kva)}{kvJ_m'(kva)}. \tag{B 3}$$

Again we can take $K^-(u) = 1$ and $K^+(u) = K(u)$.

If there is an external flow present, we have to consider the full numerator, and $K^-(u)$ is no longer unity.

REFERENCES

Bechert, D. W. 1979 Sound absorption caused by vorticity shedding demonstrated with a jet flow. *A.I.A.A. Paper* no. 79-0575.

Bechert, D. W., Michel, U. & Pfizenmaier, E. 1977 Experiments on the transmission of sound through jets. *A.I.A.A. Paper* no. 77-1928.

Bryce, W. D. 1979 Experiments concerning the anomalous behaviour of aero-engine exhaust noise in flight. *A.I.A.A. Paper* no. 79–0648.

Cargill, A. M. 1981 The aeroacoustics of jet pipes and cascades. Ph.D. thesis, University of Leeds.

CARGILL, A. M. 1982 Low frequency acoustic radiation from a jet pipe – a second order theory. *J. Sound Vib.* (in press).

CARRIER, G. F. 1956 Sound transmission from a tube with flow. *Quart. Appl. Math.* **13**, 457–461.

CRIGHTON, D. G. 1972 The excess noise field of subsonic jets. *J. Fluid Mech.* **56**, 683–694.

CUMPSTY, N. A. & MARBLE, F. E. 1977 The interaction of entropy fluctuations with turbine blade rows. *Proc. R. Soc. Lond.* A**357**, 323–344.

DOWLING, A. P., FFOWCS WILLIAMS, J. E. & GOLDSTEIN, M. E. 1978 Sound production in a moving stream. *Phil. Trans. R. Soc. Lond.* A **288**, 321–439.

FFOWCS WILLIAMS, J. E. 1972 *Current papers of the Aeronautical Research Council.* C.P. no. 1195. *Papers on novel aerodynamic source mechanisms at low jet speeds*, chap. 2.

FFOWCS WILLIAMS, J. E. & HAWKINGS, D. L. 1969 Sound generation by turbulence and surfaces in arbitrary motion. *Phil. Trans. R. Soc. Lond.* A **264**, 321–342.

FFOWCS WILLIAMS, J. E. & HOWE, M. S. 1975 The generation of sound by density inhomogeneities in low Mach number nozzle flows. *J. Fluid Mech.* **70**, 605–622.

GOLDSTEIN, M. E. 1975 The low frequency sound from multipole sources in axisymmetric shear flows with application to jet noise. *J. Fluid Mech.* **70**, 595–604.

HOWE, M. S. 1975 Contributions to the theory of aerodynamic sound with application to excess jet noise and the theory of the flute. *J. Fluid Mech.* **71**, 625–673.

HOWE, M. S. 1976 The influence of vortex shedding on the generation of sound by convected turbulence. *J. Fluid Mech.* **76**, 711–740.

HOWE, M. S. 1979 Attenuation of sound in a low Mach number nozzle flow. *J. Fluid Mech.* **91**, 209–229.

HOWE, M. S. & FFOWCS WILLIAMS, J. E. 1978 On the noise of an imperfectly expanded supersonic jet. *Phil. Trans. R. Soc.* A **289**, 271–314.

INGARD, U. & SINGHAL, V. K. 1975 Effect of flow on the acoustic resonances of an open duct. *J. Acoust. Soc. Am.* **88**, 788–793.

JACQUES, J. R. 1975 The acoustic response of a nozzle flow to an externally applied low frequency pressure field. *J. Sound Vib.* **41**, 13–32.

JONES, D. S. & MORGAN, J. D. 1974 A linear model of a finite Helmholtz instability. *Proc. R. Soc. Lond.* A **388**, 17–41.

LEPPINGTON, F. G. 1971 *Current papers of the Aeronautical Research Council.* C.P. no. 1195. *Papers on novel aerodynamic source mechanisms at low jet speeds*, chap. 5.

LEVINE, H. & SCHWINGER, J. 1948 On the radiation of sound from an unflanged circular pipe. *Phys. Rev.* **73**, 383–406.

LIGHTHILL, M. J. 1952 On sound generated aerodynamically. I. General theory. *Proc. R. Soc. Lond.* A **221**, 564–587.

MANI, R. 1973 Refraction of acoustic duct wave-guide modes by exhaust jets. *Quart. Appl. Mech.* **30**, 501–520.

MANI, R. 1974 The jet density exponent issue for the noise of heated subsonic jets. *J. Fluid Mech.* **64**, 611–622.

MANI, R. 1981 Low-frequency sound propagation in a quasi-one-dimensional flow. *J. Fluid Mech.* **104**, 81–92.

MARBLE, F. E. & CANDEL, S. M. 1977 Acoustic disturbance from gas non-uniformities convected through a nozzle. *J. Sound Vib.* **55**, 225–244.

MOORE, C. J. 1977 The role of shear layer instability waves in jet exhaust noise. *J. Fluid Mech.* **80**, 321–367.

MORGAN, J. D. 1974 The interaction of sound with a semi-infinite vortex sheet. *Quart. J. Mech. Appl. Math.* **27**, 465–487.

MUNT, R. M. 1977 The interaction of sound with a subsonic jet issuing from a semi-infinite cylindrical pipe. *J. Fluid Mech.* **83**, 609–640.

MUNT, R. M. 1982*a* Acoustic transmission properties of a jet pipe with subsonic jet flow. I. The cold jet reflection coefficient. Submitted to *J. Sound Vib.*

MUNT, R. M. 1982*b* Acoustic transmission properties of a jet pipe with subsonic jet flow. II. The cold jet radiated power. To be submitted to *J. Sound Vib.*

NOBLE, B. 1958 *Methods Based on the Wiener–Hopf Technique*, Pergamon.

PINKER, R. A. & BRYCE, W. D. 1976 The radiation of plane wave duct noise from a jet exhaust statically and in flight. *A.I.A.A. Paper* no. 76–581.

SAVKAR, S. D. 1975 Radiation of cylindrical duct acoustic modes with flow mismatch. *J. Sound Vib.* **42**, 383–386.